Problem of the Month

Problem 0: Equations in the integers

September 2025

Suppose a, b, and c are positive integers. In this problem, a *non-negative solution* to the equation $ax + by = c$ is a pair $(x, y) = (u, v)$ of integers with $u \geq 0$ and $v \geq 0$ satisfying $au + bv = c$. For example, $(x, y) = (7, 0)$ and $(x, y) = (3, 3)$ are non-negative solutions to $3x + 4y = 21$, but $(x, y) = (-1, 6)$ is not.

1. Determine all non-negative solutions to $5x + 8y = 120$.
2. Determine the largest positive integer c with the property that there is no non-negative solution to $5x + 8y = c$.

In Questions 3, 4, and 5, a and b are assumed to be positive integers satisfying $\gcd(a, b) = 1$.

3. Determine the largest non-negative integer c with the property that there is no non-negative solution to $ax + by = c$. The value of c should be expressed in terms of a and b.
4. Determine the number of non-negative integers c for which there are exactly 2025 non-negative solutions to $ax + by = c$. As with Question 3, the answer should be expressed in terms of a and b.
5. Suppose $n \geq 1$ is an integer. Determine the sum of all non-negative integers c for which there are exactly n nonnegative solutions to $ax + by = c$. The answer should be expressed in terms of a, b, and n.

Fact: You may find it useful that for integers a and b with $\gcd(a, b) = 1$, there always exist integers x and y such that $ax + by = 1$, though x and y may not be non-negative.

Problem of the Month

Problem 1: Displacing Permutations

October 2025

In this problem, X_n will denote the set of integers $\{1, 2, 3, \ldots, n\}$. A *permutation* of X_n is an ordered list of these integers that contains each of them exactly once. For example, there are exactly 6 permutations of X_3, and they are

$$123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321$$

Another way to think about a permutation is as a function from the set X_n to itself, where no two inputs to the function have the same output. For example, the permutation 213 of X_3 represents the function that sends 1 to 2, sends 2 to 1, and sends 3 to itself. The permutation σ of X_5 denoted by 43512 satisfies $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 2$.

In this problem, the *displacement* of a permutation σ of X_n is equal to $\mathbf{D}(\sigma) = \displaystyle\sum_{i=1}^{n} |i - \sigma(i)|$. For example, with σ from the paragraph above, we have

$$\begin{aligned}\mathbf{D}(\sigma) &= |1-\sigma(1)| + |2-\sigma(2)| + |3-\sigma(3)| + |4-\sigma(4)| + |5-\sigma(5)| \\ &= |1-4| + |2-3| + |3-5| + |4-1| + |5-2| \\ &= 3+1+2+3+3 = 12\end{aligned}$$

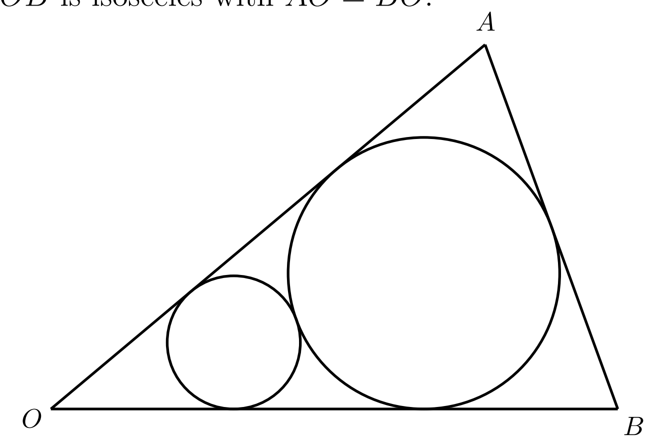
1. Suppose $n \geq 2$. Determine the number of permutations σ of X_n that satisfy $\mathbf{D}(\sigma) = 2$.
2. Suppose $n \geq 4$. Determine the number of permutations σ of X_n that satisfy $\mathbf{D}(\sigma) = 4$.
3. Prove for all $n \geq 2$, if σ is a permutation of X_n, then $\mathbf{D}(\sigma)$ is even.
4. Given an odd positive integer n and a permutation σ of X_n, determine the maximum possible value of $\mathbf{D}(\sigma)$.
5. Given an odd positive integer n, determine the number of permutations σ of X_n have the property that $\mathbf{D}(\sigma)$ is equal to the maximum from the previous question.

Problem of the Month

Problem 2: Squeezing circles into a triangle

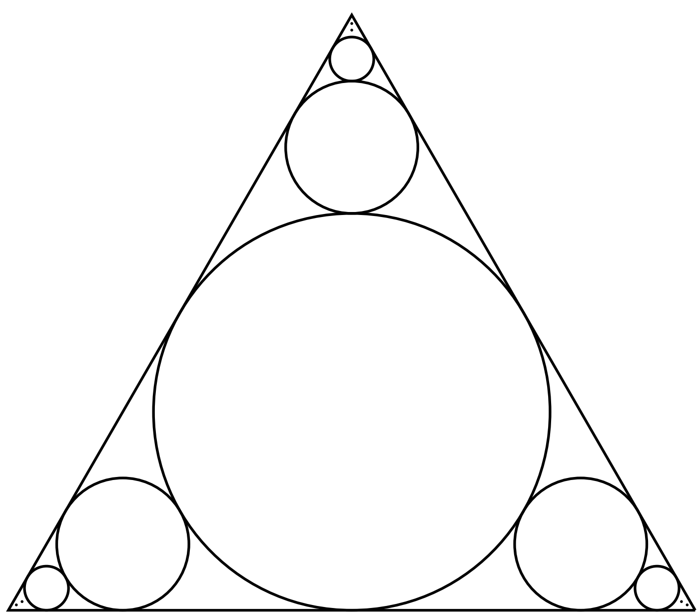
November 2025

1. Let θ be an angle with $0^\circ < \theta < 45^\circ$. In the diagram, points A and B are configured so that $\angle AOB = 2\theta$ and $\triangle AOB$ is isosceles with $AO = BO$.



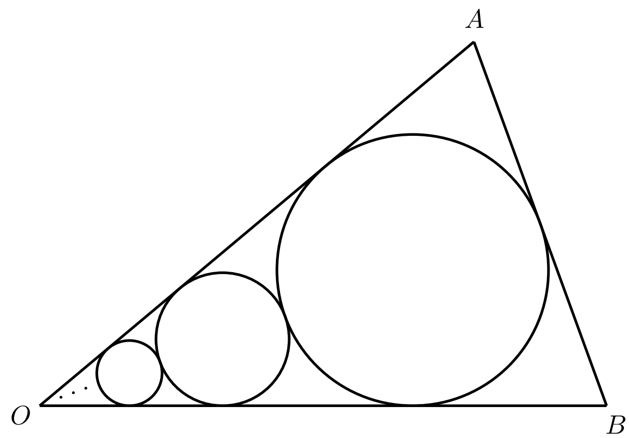
A circle is inscribed in $\triangle AOB$ and another circle is drawn so that it is tangent to the larger circle as well as OA and OB. In terms of θ, find the ratio of the radius of the larger circle to the radius of the smaller circle.

2. Similar to Question 1, an equilateral triangle has a circle inscribed in it. Three circles are then drawn, each tangent to two of the sides of the triangle as well as the larger circle. Another three circles are then drawn, each tangent to two of the three sides of the triangle as well as one of the circles drawn in the previous step.



If this process is continued indefinitely, what fraction of the area of the triangle is covered by circles?

3. Suppose $\triangle AOB$ and θ are as they were defined in Question 1. The process of drawing a circle tangent to OA, OB, and the smallest circle is repeated forever. What fraction of the area of $\triangle AOB$ is covered by circles? Your answer should be in terms of θ.



The result of Question 3 can be applied to solve Question 2. Can you see how?

Problem of the Month

Problem 3: Multiplication in two dimensions

December 2025

This month's problem is inspired by Question B2(c) on the 2025 Canadian Senior Mathematics Contest. Here is the original question:

> The points X, Y, and Z have coordinates $X(0,0)$, $Y(7,24)$, and $Z(15,0)$. The point W is on the line segment YZ such that $\angle WXZ = 3\angle WXY$. Determine the coordinates of W.

Give it a try as a warm up before reading on!

One path to solving this problem is to introduce a point V on the line segment YZ so that $\angle VXZ = \angle VXY$. If you do this, you find the coordinates of V are $(12,9)$. These are very nice numbers! In this month's POTM, we will investigate how the points X, Y, and Z were chosen so that the coordinates of V (and eventually W) are nice rational numbers. This will involve a way to multiply points on the Cartesian plane.

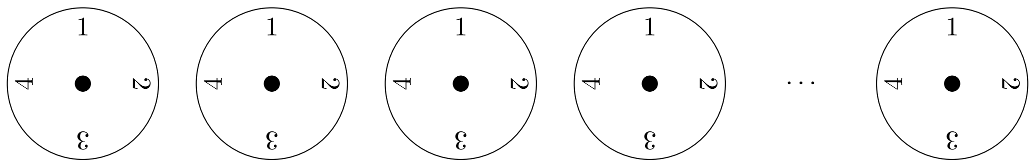
1. Given points (a,b) and (c,d) on the Cartesian plane, define their *product* as
$$(a,b) * (c,d) = (ac - bd, ad + bc).$$
So, for example, $(3,5) * (2,-1) = (3 \cdot 2 - 5 \cdot (-1), 3 \cdot (-1) + 5 \cdot 2) = (11,7)$.
   (a) Find a point (a,b) satisfying $(1,1) * (a,b) = (0,-2)$.
   (b) Find a point (a,b) with the property that $(a,b) * (c,d) = (c,d)$ for every point (c,d) in the Cartesian plane.
2. For a point (a,b) in the Cartesian plane, denote by $(a,b)^k$ the point obtained by taking the product of (a,b) with itself k times. Compute $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^{2025}$.
3. Denote by O the origin $(0,0)$, and by E the point with coordinates $(1,0)$. Let A be a point in the Cartesian plane. Define $|A|$ to be the distance from A to O. Define $\theta(A)$ to be the measure of the angle $\angle EOA$, measured counterclockwise around O from the line segment OE. For example, $|(-\sqrt{2},-\sqrt{2})| = \sqrt{(-2)^2 + (-2)^2} = 2$ and $\theta(-\sqrt{2},-\sqrt{2}) = 225^\circ$.
   (a) Compute $|(0,2) * (1,1)|$ and $\theta((0,2) * (1,1))$.
   (b) Let D_1 and D_2 be points with $|D_1| = r_1$, $|D_2| = r_2$, $\theta(D_1) = \phi_1$ and $\theta(D_2) = \phi_2$. Compute $|D_1 * D_2|$ and $\theta(D_1 * D_2)$ in terms of r_1, r_2, ϕ_1, and ϕ_2.
4. The point $(2,3)$ satisfies the equation $x^4 = (-119,-120)$ since $(2,3)^4 = (-119,-120)$. Find three other points satisfying $x^4 = (-119,-120)$.
5. Let Y have coordinates $(7,24)$. Find a point F in the Cartesian plane with integer coordinates so that $OF^2 = OY$ and $2\angle EOF = \angle EOY$.

Problem of the Month

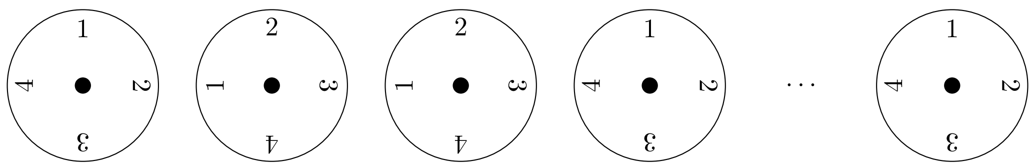
Problem 4: Rotating dials

January 2026

1. Several dials are each labelled with the integers from 1 through 4 in clockwise order. The dials are arranged in a row and initially configured so that each dial shows 1 at the top:



A *move* consists of rotating two adjacent dials in the same direction by the same number of positions. For example, one possible move is to rotate the second and third dials by one position in the counterclockwise direction resulting in the configuration shown:



There are $k \geq 2$ dials and they are initially configured so that each shows 1 at the top. In terms of k, determine how many possible configurations of the dials are attainable by performing a sequence of moves.

2. Suppose now that there is an integer $n \geq 2$ so that each of the $k \geq 2$ dials is labelled in clockwise order by the integers 1 through n beginning with 1 at the top. Find the number of configurations of the dials that are attainable by a sequence of moves. Your answer should be in terms of n and k. Again, a move consists of rotating two adjacent dials in the same direction by the same number of positions.

3. Answer Question 2 with the following additional type of move allowed: rotate the leftmost and rightmost dials in the same direction by the same number of positions.

Problem of the Month

Problem 5: Stabilising Sequences

February 2026

Suppose you have an infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers. We can consider the differences between consecutive terms in the sequence to get a new sequence. We can then consider differences between terms in the new sequence, to get yet another sequence, and so on! To keep track of all of this, for all integers $n \geq 0$ and $k \geq 0$ define

$$a_n^{(0)} = a_n \quad \text{and} \quad a_n^{(k)} = a_{n+1}^{(k-1)} - a_n^{(k-1)}.$$

For example, if the original sequence is given by $a_n = n^2$, then

$$\begin{aligned}(a_0, a_1, a_2, a_3, \ldots) = \left(a_0^{(0)}, a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, \ldots\right) &= (0, 1, 4, 9, \ldots)\\ \left(a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots\right) &= (1, 3, 5, 7, \ldots)\\ \left(a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots\right) &= (2, 2, 2, 2, \ldots).\end{aligned}$$

Note that the superscripts (for example the (2) in $a_1^{(2)}$) are *not* exponents, but just notation to keep track of everything.

We say a sequence $a_0, a_1, a_2, \ldots$ *stabilises at layer* k if k is the smallest integer for which $a_n^{(k)} = a_{n+1}^{(k)}$ for all $n \geq 0$. For example, the sequence $a_0, a_1, a_2, \ldots$ defined by $a_n = n^2$ stabilises at layer 2.

1. A sequence that stabilises at layer 1 begins with $a_0 = 5$ and $a_1 = 8$. Compute a_{2026}.
2. The sequence $a_0, a_1, a_2, \ldots$ defined by $a_n = 7n^4$ stabilises at layer k. Find k and compute $a_n^{(k)}$ for every integer $n \geq 0$.
3. A sequence $a_0, a_1, a_2, \ldots$ is defined by $a_n = C_t n^t + C_{t-1} n^{t-1} + C_{t-2} n^{t-2} + \cdots + C_1 n + C_0$ for some integer $t \geq 0$ and some constant real numbers $C_0, C_1, \ldots, C_t$ with $C_t \neq 0$. Show that the sequence stabilises at layer k for some k, and compute $a_n^{(k)}$ for every integer $n \geq 0$.
4. A sequence that stabilises at layer 3 begins with $a_0 = 7$, $a_1 = 5$, $a_2 = 13$, and $a_3 = 43$. Compute a_{2026}.

Problem of the Month
Problem 6: Cobbling cards

March 2026

The card game Cobble (not to be confused with the popular card game Dobble, also known as Spot It) is played with a special deck consisting of finitely many cards, each containing finitely many symbols. A Cobble deck satisfies the following properties:

(i) Each symbol appears at most once on each card.

(ii) Each pair of cards has exactly one symbol in common.

(iii) For any pair of symbols, there is exactly one card on which both symbols appear.

(iv) In the deck, there is some set of four symbols with the property that no three of them appear on the same card.

We denote each card by square braces, with the symbols listed inside. For example, $[A, B, D]$ denotes the card that contains the symbols A, B, and D. The collection of three cards

$$[A, B], [A, C], [B, C]$$

satisfies Properties (i), (ii), and (iii), but not Property (iv). Therefore, it is *not* a Cobble deck.

1. Construct a Cobble deck with seven cards using the seven symbols $\{A, B, C, D, E, F, G\}$.
2. Prove that there does not exist a Cobble deck with fewer than seven cards.
3. Show that in a Cobble deck, any two cards must each contain the same number of symbols. Conclude that each cord in a Cobble deck contains the same number of symbols.
4. Show that in a Cobble deck, for any two distinct symbols, they must appear on the same number of cards, and that number is the same as the number from Question 3. Conclude that each symbol in a Cobble deck appears on the same number of cards.
5. Prove that you cannot have a Cobble deck with 2026 cards.

Challenge. Construct a Cobble deck with 157 cards, or prove that you cannot do so.

Problem of the Month

Problem 7: Sneaky Values

April 2026

The denominations that make up the currency in Canada, ignoring those less than a dollar, are \$1, \$2, \$5, \$10, \$20, \$50, and \$100. Using these denominations, we can realise any integer dollar value. For example, to make \$45, we could use four \$10s and one \$5, or two \$20s and one \$5, or even forty-five \$1s. The way to realise \$45 that uses the fewest items (coins or notes) is with two \$20s and one \$5, which uses three items.

1. Describe, with justification, how to obtain \$107 in Canada using the least number of items.
2. In the utopian (and sadly, fictitious) country of Cemcalia, the currency has notes with denominations 1, 5, 10, 18, and 30. Describe, with justification, how to obtain 37 using the least number of notes.

In order to investigate general currency systems, we will first set some notation. A *denomination set* is a tuple $(d_1, d_2, \ldots, d_k)$ of positive integers satisfying $d_1 = 1$ and $d_i < d_{i+1}$ for all i. Given such a denomination set, a *realisation* of a value v (which must be a positive integer) is a k-tuple of non-negative integers $N = (n_1, n_2, \ldots, n_k)$ so that $n_1 d_1 + n_2 d_2 + \cdots + n_k d_k = v$. Denote the number of items used in the realisation N by $|N| = n_1 + n_2 + \cdots + n_k$. The *greedy realisation* of v (denoted $G(v)$) is obtained by taking the largest denominations possible until the value is reached.

For example, in Canada, the denomination set is the 7-tuple $(1, 2, 5, 10, 20, 50, 100)$. The greedy realisation of the value 45 is $G(45) = (0, 0, 1, 0, 2, 0, 0)$ and $|G(45)| = 1 + 2 = 3$. The value 45 is also realised by the 7-tuples $(0, 0, 1, 4, 0, 0, 0)$ and $(45, 0, 0, 0, 0, 0, 0)$ (among others).

It is not necessarily the case that the greedy realisation uses the fewest items! For a value v, define $E(v)$ to be the smallest possible number $|N|$ over all realisations N of v. For example, consider the denomination set $(1, 15, 20)$. Then $E(30) = 2$ and $|G(30)| = |(10, 0, 1)| = 11$. It is a consequence of the definition of $E(v)$ that $E(v) \leq |G(v)|$ for all values v. A value v for which $E(v) < |G(v)|$ is called a *sneaky value*.

3. Find infinitely many denomination sets $(1, d_2, d_3)$ with the property that
   (a) $d_3 + 2$ is the smallest sneaky value.
   (b) $d_2 + d_3 - 1$ is the smallest sneaky value.
4. Let $(d_1, d_2, d_3, \ldots, d_k)$ be a denomination set.
   (a) Show that for any value $v > d_k$, $|G(v - d_k)| + 1 = |G(v)|$.
   (b) For all values v and all denominations $d_i < v$, show that $E(v - d_i) + 1 \geq E(v)$. When is the inequality an equality?
   (c) Suppose there are no sneaky values v satisfying $d_3 + 1 < v < d_{k-1} + d_k$. Prove that there are no sneaky values at all!

5. Using your favourite programming language, write a program that decides whether or not a denomination set $(d_1, d_2, \ldots, d_k)$ admits a sneaky value.

Problem of the Month

Problem 8: Revolutionary counting

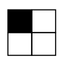
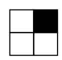
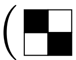
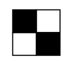
May 2026

This Problem of the Month is inspired by Question 24 of the Team Problems from the CEMC's 2026 Canadian Team Mathematics Contest:

> In a 4×4 grid consisting of 16 unit squares, 4 squares are coloured red, while the other 12 are coloured white. Two colourings are said to be equivalent if one can be obtained from the other by doing a sequence of 90° rotations. How many inequivalent colourings are there?

Try the problem as a warm up! The following problems will investigate counting problems of the form "in how many inequivalent ways can you colour something where two colourings are equivalent if one can be obtained from the other by a rotation?"

1. Consider a 2×2 grid of four unit squares, where each square is coloured either black or white. Let r_1, r_2, r_3, and r_0 denote the rotations of the grid by 90° clockwise, 180° clockwise, 270° clockwise, and 0° respectively. View these four rotations as functions that take as an input a colouring of the grid, and give as an output a colouring. For example,

   $r_1($  $) =$ , $\quad r_2($  $) =$ ,

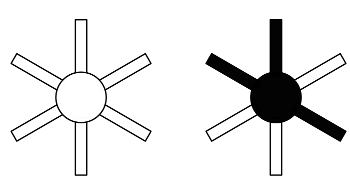
   and $r_0(c) = c$ for all colourings c. Two colourings c and d are equivalent if there is a rotation r so that $r(c) = d$.

   (a) How many inequivalent colourings of the 2×2 grid exist?

   (b) For each rotation r, let $\text{fix}(r)$ be the number of colourings c satisfying $r(c) = c$. Compute $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3)$.

   (c) Compare your answers to the previous two parts.

2. Consider a windmill with six identical blades evenly spaced around a central circle. The central circle, and each of the blades, are to be coloured either black or white. Let r_0, r_1, r_2, r_3, r_4, and r_5 denote the rotations of the windmill by $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ$, and 300° clockwise, respectively. Two colourings c and d are equivalent if there is a rotation r such that $r(c) = d$. The image below shows two inequivalent colourings of the windmill.

   (a) How many inequivalent colourings of the windmill are there?

   (b) For each rotation r let $\text{fix}(r)$ denote the number of colourings c of the windmill satisfying $r(c) = c$. Compute $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3) + \text{fix}(r_4) + \text{fix}(r_5)$.

(c) Compare your answers to the previous two parts.

3. Now for a more general approach! Suppose we have some object and k rotations $r_0, r_1, \ldots, r_{k-1}$ we can perform on the object, where r_z is a rotation by $\frac{360z}{k}$ degrees clockwise. Assume that the object can be coloured according to some rules, and consider the collection of all possible colourings of the object according to the rules. In Question 1 for example, the object is the 2×2 grid, $k = 4$ since there are four rotations, and there are $2^4 = 16$ possible colourings of the grid where each square is either black or white. Two colourings c and d are *equivalent* if there is a rotation r so that $r(c) = d$. Let N be the number of inequivalent colourings of the object.

   (a) Fix a colouring c. Let $E(c)$ be the number of colourings (including c itself) that are equivalent to c. Let $S(c)$ be the number of rotations satisfying $r(c) = c$. Show that $E(c)S(c) = k$.

   (b) A *stable pair* is a pair (r, c) where r is a rotation, c is a colouring, and $r(c) = c$. Show that the number of stable pairs is equal to kN.

   (c) For a rotation r, let $\text{fix}(r)$ denote the number of colourings c satisfying $r(c) = c$. Show that the number of stable pairs is equal to $\text{fix}(r_0) + \text{fix}(r_1) + \cdots + \text{fix}(r_{k-1})$.

4. Consider the 4×4 grid of 16 unit squares given at the beginning of this document. Let r_0, r_1, r_2, and r_3 be the rotations by $0°$, $90°$, $180°$, and $270°$ clockwise, respectively. By computing $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3)$, solve the problem at the beginning of this document.

5. Let p be a prime. Consider a windmill with p identical blades evenly spaced around a central circle. The central circle, and each of the blades, are to be coloured either black or white. Two colourings are considered equivalent if you can rotate one into the other. How many inequivalent colourings of the windmill are there?