

Problem of the Week
Problem E and Solution
Random Number Threads 3

Problem

Travis creates t-shirts printed with three random numbers. The first number is a random integer from 1 to 20, inclusive, printed in green. The second number is a random integer from 1 to 12, inclusive, printed in pink. The third number is a random integer from 1 to 8, inclusive, printed in blue.

One of Travis' t-shirts is randomly chosen. What is the probability that the sum of the three random numbers on this t-shirt is divisible by 13?

Solution

There are 20 different possibilities for the green number, 12 different possibilities for the pink number, and 8 different possibilities for the blue number. That means there is a total of $20 \times 12 \times 8 = 1920$ different t-shirts that could be printed.

Let (g, p, b) represent the numbers on a t-shirt where g is the green number, p is the pink number, and b is the blue number. Also, $1 \leq g \leq 20$, $1 \leq p \leq 12$, and $1 \leq b \leq 8$, for integers g, p, b.

The minimum possible sum on a t-shirt is $1 + 1 + 1 = 3$ and the maximum possible sum is $20 + 12 + 8 = 40$. The only multiples of 13 within this range are 13, 26, and 39. We will consider cases for each of these sums. For the first two cases we will consider sub-cases based on the value of the blue number.

- Case 1: The sum of the numbers is 13.
  - The blue number is 1.
    Then the sum of the other two numbers is 12. The possibilities for (g, p, b) are $(1, 11, 1)$, $(2, 10, 1)$, $(3, 9, 1)$, ..., $(11, 1, 1)$, which is 11 possibilities in total.
  - The blue number is 2.
    Then the sum of the other two numbers is 11. The possibilities for (g, p, b) are $(1, 10, 2)$, $(2, 9, 2)$, $(3, 8, 2)$, ..., $(10, 1, 2)$, which is 10 possibilities in total.
  - The blue number is 3.
    Then the sum of the other two numbers is 10. The possibilities for (g, p, b) are $(1, 9, 3)$, $(2, 8, 3)$, $(3, 7, 3)$, ..., $(9, 1, 3)$, which is 9 possibilities in total.
  - The blue number is 4.
    Then the sum of the other two numbers is 9. The possibilities for (g, p, b) are $(1, 8, 4)$, $(2, 7, 4)$, $(3, 6, 4)$, ..., $(8, 1, 4)$, which is 8 possibilities in total.
  - The blue number is 5.
    Then the sum of the other two numbers is 8. The possibilities for (g, p, b) are $(1, 7, 5)$, $(2, 6, 5)$, $(3, 5, 5)$, ..., $(7, 1, 5)$, which is 7 possibilities in total.
  - The blue number is 6.
    Then the sum of the other two numbers is 7. The possibilities for (g, p, b) are $(1, 6, 6)$, $(2, 5, 6)$, $(3, 4, 6)$, ..., $(6, 1, 6)$, which is 6 possibilities in total.

- The blue number is 7.
  Then the sum of the other two numbers is 6. The possibilities for (g, p, b) are $(1, 5, 7)$, $(2, 4, 7)$, $(3, 3, 7)$, ..., $(5, 1, 7)$, which is 5 possibilities in total.
- The blue number is 8.
  Then the sum of the other two numbers is 5. The possibilities for (g, p, b) are $(1, 4, 8)$, $(2, 3, 8)$, $(3, 2, 8)$, $(4, 1, 8)$, which is 4 possibilities in total.

In total, there are $11 + 10 + 9 + \cdots + 5 + 4 = 60$ possibilities with a sum of 13.

- Case 2: The sum of the numbers is 26.
  - The blue number is 1.
    Then the sum of the other two numbers is 25. Remember that the largest green number is 20 and the largest pink number is 12. The possibilities for (g, p, b) are $(20, 5, 1)$, $(19, 6, 1)$, $(18, 7, 1)$, ..., $(13, 12, 1)$, which is 8 possibilities in total.
  - The blue number is 2.
    Then the sum of the other two numbers is 24. The possibilities for (g, p, b) are $(20, 4, 2)$, $(19, 5, 2)$, $(18, 6, 2)$, ..., $(12, 12, 2)$, which is 9 possibilities in total.
  - The blue number is 3.
    Then the sum of the other two numbers is 23. The possibilities for (g, p, b) are $(20, 3, 3)$, $(19, 4, 3)$, $(18, 5, 3)$, ..., $(11, 12, 3)$, which is 10 possibilities in total.
  - The blue number is 4.
    Then the sum of the other two numbers is 22. The possibilities for (g, p, b) are $(20, 2, 4)$, $(19, 3, 4)$, $(18, 4, 4)$, ..., $(10, 12, 4)$, which is 11 possibilities in total.
  - The blue number is 5.
    Then the sum of the other two numbers is 21. The possibilities for (g, p, b) are $(20, 1, 5)$, $(19, 2, 5)$, $(18, 3, 5)$, ..., $(9, 12, 5)$, which is 12 possibilities in total.
  - The blue number is 6.
    Then the sum of the other two numbers is 20. The possibilities for (g, p, b) are $(19, 1, 6)$, $(18, 2, 6)$, $(17, 3, 6)$, ..., $(8, 12, 6)$, which is 12 possibilities in total.
  - The blue number is 7.
    Then the sum of the other two numbers is 19. The possibilities for (g, p, b) are $(18, 1, 7)$, $(17, 2, 7)$, $(16, 3, 7)$, ..., $(7, 12, 7)$, which is 12 possibilities in total.
  - The blue number is 8.
    Then the sum of the other two numbers is 18. The possibilities for (g, p, b) are $(17, 1, 8)$, $(16, 2, 8)$, $(15, 3, 8)$, ..., $(6, 12, 8)$, which is 12 possibilities in total.

  In total, there are $8 + 9 + 10 + 11 + 12(4) = 86$ possibilities with a sum of 26.

- Case 3: The sum of the numbers is 39.
  Since the maximum possible sum is 40, in order to get a sum of 39, two of the numbers will be at their maximum, and the other number will be one less than its maximum. Thus, the possibilities for (g, p, b) are $(20, 12, 7)$, $(20, 11, 8)$, and $(19, 12, 8)$, which is 3 possibilities in total.

The total number of possible sums that are divisible by 13 is $60 + 86 + 3 = 149$. The probability of randomly selecting a t-shirt whose numbers sum to a multiple of 13 is then $\frac{149}{1920}$.