

Problem of the Week
Problem E and Solution
Competition for Cache

Problem

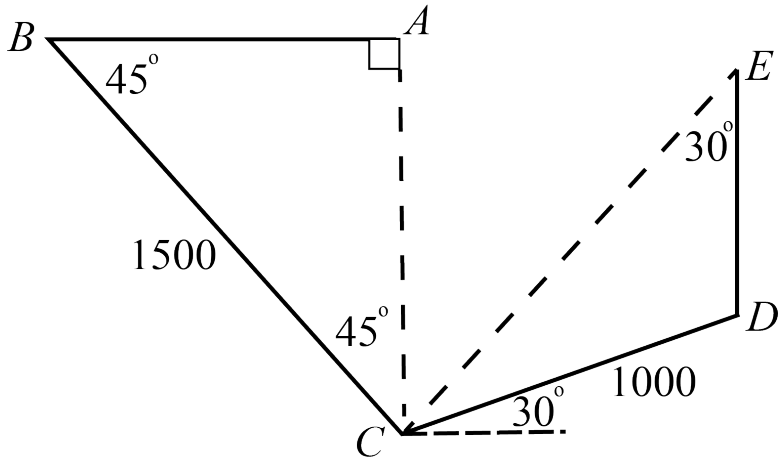
A Geocaching race organizer is putting four caches along a route that starts and finishes at the same place.

While making his route, he recorded a few route details. Starting from point A, the organizer travelled due west to point B. From point B, he travelled 1500 m southeast to point C. While at C, he noted that $\angle BCA$ is $45°$. From point C, he travelled 1000 m in a direction $30°$ north from east to point D. Finally from point D, he travelled due north to point E and noted that $\angle DEC = 30°$.

Determine the distance, rounded to the nearest metre, from the cache at point E to the race finish at point A.

Solution

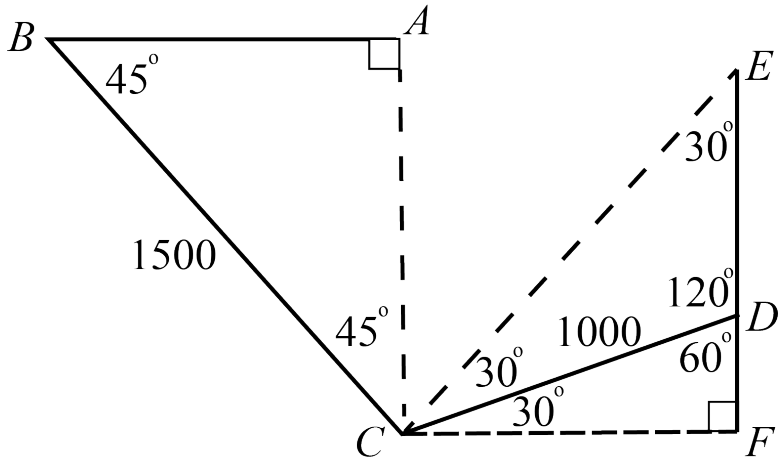
Since he travelled west from A to B then southeast from B to C, we know that $\angle ABC = 45°$. We are given that $\angle BCA = 45°$. Thus, $\angle BAC = 90°$, and so $\triangle ABC$ is isosceles with $AB = AC$ and $BC = 1500$.



In $\triangle ABC$, we can use the sine ratio to solve for length AC:

$$\sin(45°) = \frac{AC}{BC}$$
$$\frac{1}{\sqrt{2}} = \frac{AC}{1500}$$
$$AC = \frac{1500}{\sqrt{2}} = 750\sqrt{2}$$

Extend ED to F so that $\angle CFD = 90°$. In $\triangle CDE$, we are given $\angle DEC = 30°$. Since $\angle CDF = 60°$, we have $\angle CDE = 180° - 60° = 120°$, and therefore $\angle DCE = 180° - 120° - 30° = 30°$.



We can now determine CE using the sine law:

$$\begin{aligned}
\frac{CE}{\sin(120°)} &= \frac{1000}{\sin(30°)} \\
CE &= \frac{1000\sin(120°)}{\sin(30°)} \\
&= \frac{1000\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}} \\
&= 1000\sqrt{3}
\end{aligned}$$

Now, $\angle ACE = 90° - (30° + 30°) = 30°$.

Finally, knowing the lengths AC and CE, along with $\angle ACE$, we can apply the cosine law to $\triangle ACE$, to determine AE:

$$\begin{aligned}
AE^2 &= AC^2 + CE^2 - 2(AC)(CE)\cos(\angle ACE) \\
&= (750\sqrt{2})^2 + (1000\sqrt{3})^2 - 2(750\sqrt{2})(1000\sqrt{3})\cos(30°) \\
&= 1\,125\,000 + 3\,000\,000 - 2(750\sqrt{2})(1000\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) \\
&= 4\,125\,000 - (750\sqrt{2})(1000)(3) \\
&= 4\,125\,000 - 2\,250\,000\sqrt{2}
\end{aligned}$$

Therefore the distance, from the final cache location to the finish line, is $\sqrt{4\,125\,000 - 2\,250\,000\sqrt{2}} \approx 971$ m.