Problem of the Week

Problems and Solutions 2025-2026

Problem E

Grade 11/12

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Table of Contents

The problems in this booklet are organized into themes.
A problem often appears in multiple themes.
Click on the theme name to jump to that section.

Algebra (A)

Take me to the cover

Problem of the Week
Problem E
Sticky Notes

The students in an art class displayed their final projects for an art exhibit at their school. Some of the projects were set up in the gym and the rest were in the library. After the exhibit finished, each student in the class went around and put a sticky note with a positive comment on each of their classmate's projects. Once they were done, the projects in the gym had a total of 308 sticky notes and the projects in the library had a total of 198 sticky notes. How many students were in the art class? How many of their projects were in the library?

Problem of the Week

Problem E and Solution

Sticky Notes

Problem

The students in an art class displayed their final projects for an art exhibit at their school. Some of the projects were set up in the gym and the rest were in the library. After the exhibit finished, each student in the class went around and put a sticky note with a positive comment on each of their classmate's projects. Once they were done, the projects in the gym had a total of 308 sticky notes and the projects in the library had a total of 198 sticky notes. How many students were in the art class? How many of their projects were in the library?

Solution

Let n represent the number of students in the art class. Each student wrote $n-1$ sticky notes, so in total there were $n(n-1)$ sticky notes. We know the total number of sticky notes is $308 + 198 = 506$. Thus,

$$\begin{aligned} n(n-1) &= 506 \\ n^2 - n &= 506 \\ n^2 - n - 506 &= 0 \\ (n-23)(n+22) &= 0 \end{aligned}$$

Therefore $n = 23$ or $n = -22$. Since n represents the number of students, it follows that $n > 0$, so we can conclude that $n = 23$. Then there were 23 students in the class and they each wrote 22 sticky notes for their classmates' projects and each received 22 sticky notes on their own project.

Since there were a total of 198 sticky notes in the library, and each project had 22 sticky notes, it follows that there were $198 \div 22 = 9$ projects in the library.

Therefore, there were 23 students in the art class and 9 of their projects were in the library.

Problem of the Week
Problem E
To the Peak with Powers

Avital can choose four different numbers w, x, y, and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

$$w^x + y^z$$

$w^x + y^z$

Problem of the Week
Problem E and Solution
To the Peak with Powers

Problem

Avital can choose four different numbers w, x, y, and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

Solution

Consider w^x, where w and x are different numbers from $\{-1, -2, -3, -4, -5\}$. What is the largest possible value for w^x?

Since x will be negative, we write $w^x = \frac{1}{w^{-x}}$, where $-x > 0$.

If x is odd, then since w is negative, then w^x will be negative.
If x is even, then w^x will be positive.

So to make w^x as large as possible, we make x even. So $x = -2$ or $x = -4$.

Also, in order to make $w^x = \frac{1}{w^{-x}}$ as large as possible, we want to make the denominator, w^{-x}, as small as possible, so w should be as small as possible in absolute value.

Therefore, the largest possible value of w^x will be when $w = -1$ and $x = -2$ or $x = -4$. In both cases, $w^x = 1$, since $(-1)^{-2} = (-1)^{-4} = 1$.

What is the second largest possible value for w^x?

Again, we need x to be even to make w^x positive, and from above, we can assume that $w \neq -1$.

When $x = -2$, the smallest possible base (in absolute value) is $w = -3$ and $w^x = \frac{1}{(-3)^2} = \frac{1}{9}$.

When $x = -4$, the smallest possible base (in absolute value) is $w = -2$ and $w^x = \frac{1}{(-2)^4} = \frac{1}{16}$.

The largest of these two values for w^x is $\frac{1}{9}$.

Therefore, the two largest possible values for w^x are 1 and $\frac{1}{9}$.

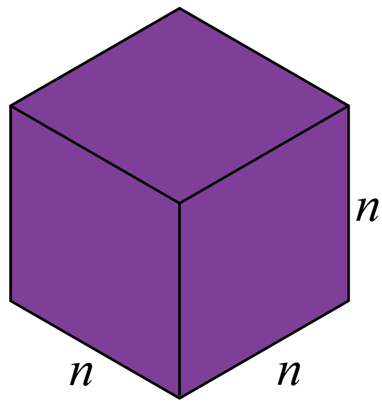
Thus, the largest possible value of $w^x + y^z$ is $1 + \frac{1}{9} = \frac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$. This will occur when $w = -1$, $x = -4$, $y = -3$, and $z = -2$ or when $w = -3$, $x = -2$, $y = -1$, and $z = -4$.

Problem of the Week

Problem E

Painting Faces

A cube has side length n, where n is a positive integer. Exactly three faces, which meet at a corner, are painted purple. The cube is then cut into n^3 smaller cubes with side length 1. If exactly 125 of these smaller cubes have no faces painted purple, then determine the value of n.



Problem of the Week

Problem E and Solution

Painting Faces

Problem

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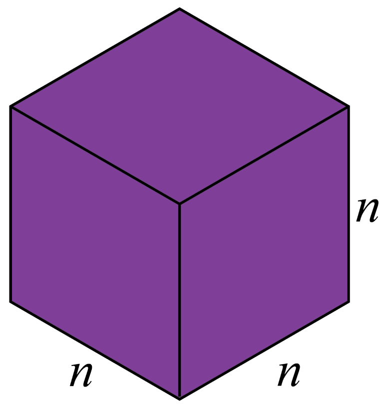
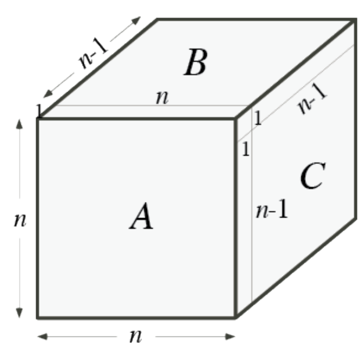


Solution

Solution 1

We label the faces painted purple as A, B, and C. We know that there are n^3 smaller cubes of side length 1. To determine the number of smaller cubes with all faces unpainted, we can subtract the number of cubes with some purple faces from the total number of cubes. Face A has dimensions n by n, and so contains n^2 unit cubes with some purple. Ignoring the area on face B which includes the smaller cubes also on face A, the remainder of face B has dimensions n by $(n-1)$, and so contains $n \times (n-1)$ more unit cubes with some purple. Ignoring the area on face C which includes the smaller cubes also on face A or on face B, face C has dimensions $(n-1)$ by $(n-1)$, and so contains $(n-1)(n-1)$ more unit cubes with some purple.



The number of unpainted cubes is thus, $n^3 - n^2 - n(n-1) - (n-1)(n-1)$. We can simplify this as

$$n^3 - n^2 - n(n-1) - (n-1)(n-1) = n^2(n-1) - n(n-1) - (n-1)(n-1)$$

Each term in this expression contains a common factor of $(n-1)$, so the expression simplifies to $(n-1)(n^2-n-(n-1)) = (n-1)(n^2-2n+1)$. This further simplifies to $(n-1)^3$. If the solver pauses here to think about this, if the unit cubes on face A then face B and finally face C are removed, we are left with a cube with side length $(n-1)$, and so $(n-1)^3$ unit cubes.

So, $(n-1)^3 = 125$, the actual number of unpainted cubes.

Taking the cube root, $n-1=5$, and so $n=6$ follows.

Solution 2

This solution uses the factor theorem which is generally taught in Grade 12.

As in Solution 1, the number of unpainted cubes is $n^3-n^2-n(n-1)-(n-1)(n-1)$. Setting this equal to 125, we have

$$\begin{aligned} n^3-n^2-n(n-1)-(n-1)(n-1) &= 125 \\ n^3-n^2-n^2+n-n^2+2n-1 &= 125 \\ n^3-3n^2+3n-126 &= 0 \end{aligned}$$

Let $f(n) = n^3-3n^2+3n-126$.

When $n=6$, $f(6) = 6^3-3(6^2)+3(6)-126 = 216-108+18-126 = 224-224 = 0$.

Since $f(6)=0$, $(n-6)$ is a factor of $f(n)$.

After long division (or synthetic division), we find $f(n) = (n-6)(n^2+3n+21)$.

Since $n^2+3n+21=0$ has no real solution, we know that $n=6$ is the only real solution to $n^3-3n^2+3n-126=0$.

Therefore, $n=6$.

Problem of the Week

Problem E

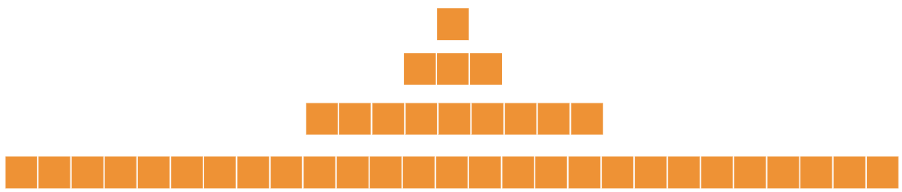
Power Trail

The first four terms of a geometric sequence are

$$a,\ b,\ 4a,\ a-6+b$$

for some numbers a and b with $a \neq 0$.

Determine all possibilities for the values of a and b.



NOTE: A geometric sequence is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 2, 6, 18 is a geometric sequence with three terms and a common ratio of 3. You can explore geometric sequences further in our CEMC Courseware.

Problem of the Week

Problem E and Solution

Power Trail

Problem

The first four terms of a geometric sequence are

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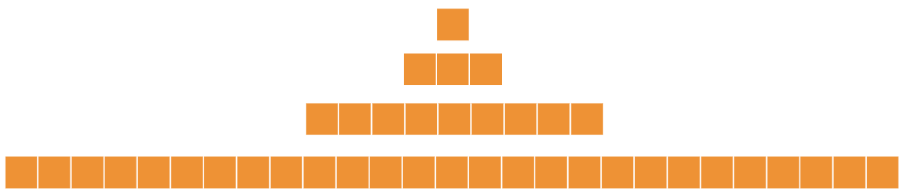
Determine all possibilities for the values of a and b.



NOTE: A geometric sequence is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 2, 6, 18 is a geometric sequence with three terms and a common ratio of 3. You can explore geometric sequences further in our CEMC Courseware.

Solution

In a geometric sequence with first term t_1, second term t_2, third term t_3, and fourth term t_4, we have $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3} = \dfrac{t_3}{t_4}$.

Here, $t_1 = a$, $t_2 = b$, $t_3 = 4a$, and $t_4 = a - 6 + b$.

Therefore, from $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3}$, we have

$$\begin{aligned} \frac{a}{b} &= \frac{b}{4a} \\ 4a^2 &= b^2 \\ 2a &= \pm b \end{aligned}$$

There are now two cases to consider.

- **Case 1:** When $b = 2a$, then the sequence can be written as a, $2a$, $4a$, $3a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned}\frac{a}{2a} &= \frac{4a}{3a-6}\\ 3a^2 - 6a &= 8a^2\\ 0 &= 5a^2 + 6a\\ &= a(5a+6)\end{aligned}$$

  Thus, $a = 0$ or $a = -\frac{6}{5}$. Since $a \neq 0$, we have $a = -\frac{6}{5}$.
  Then $b = 2a = 2(-\frac{6}{5}) = -\frac{12}{5}$ and $a - 6 + b = -\frac{6}{5} - 6 + (-\frac{12}{5}) = -\frac{48}{5}$
  Therefore, the sequence, a, b, $4a$, $a - 6 + b$ is $-\frac{6}{5}, -\frac{12}{5}, -\frac{24}{5}, -\frac{48}{5}$.
  This sequence is indeed geometric, with first term $-\frac{6}{5}$ and common ratio 2.

- **Case 2:** When $b = -2a$, then the sequence can be written as a, $-2a$, $4a$, $-a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned}\frac{a}{-2a} &= \frac{4a}{-a-6}\\ -a^2 - 6a &= -8a^2\\ 7a^2 - 6a &= 0\\ a(7a-6) &= 0\end{aligned}$$

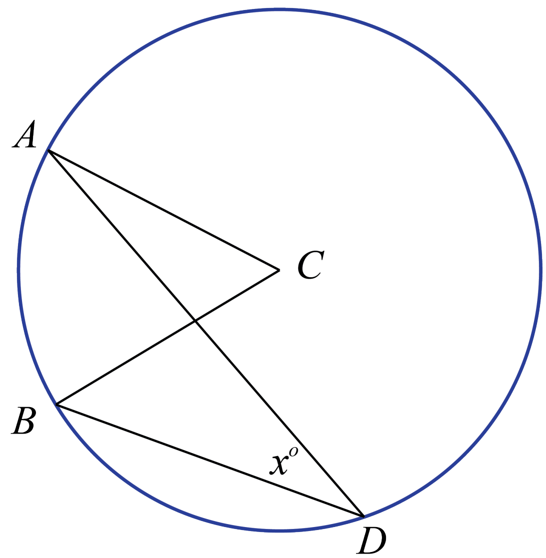
  Thus, $a = 0$ or $a = \frac{6}{7}$. Since $a \neq 0$, we have $a = \frac{6}{7}$.
  Then $b = -2a = -2(\frac{6}{7}) = -\frac{12}{7}$ and $a - 6 + b = \frac{6}{7} - 6 + (-\frac{12}{7}) = -\frac{48}{7}$
  Therefore, the sequence, $a, b, 4a, a - 6 + b$ is $\frac{6}{7}, -\frac{12}{7}, \frac{24}{7}, -\frac{48}{7}$.
  This sequence is indeed geometric, with first term $\frac{6}{7}$ and common ratio -2.

Therefore, there are two possibilities. It could be the case that $a = -\frac{6}{5}$ and $b = -\frac{12}{5}$, or it could be the case that $a = \frac{6}{7}$ and $b = -\frac{12}{7}$.

Problem of the Week
Problem E
Slice of an Arc

Points A, B, and D lie on the circumference of a circle with centre C.

If $\angle ADB = x^\circ$, then determine the measure of $\angle ACB$ in terms of x.



Problem of the Week
Problem E and Solution
Slice of an Arc

Problem

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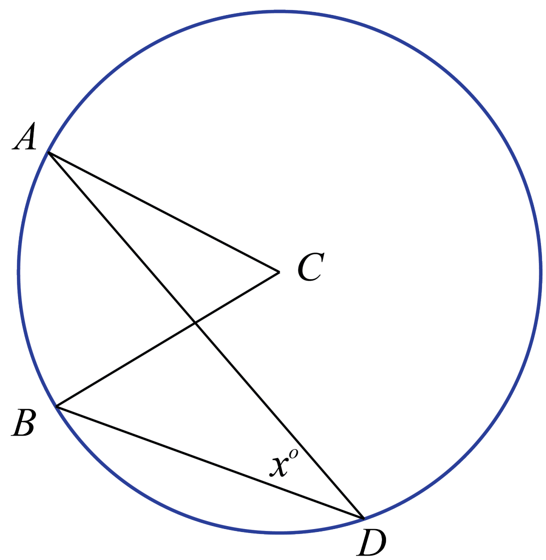
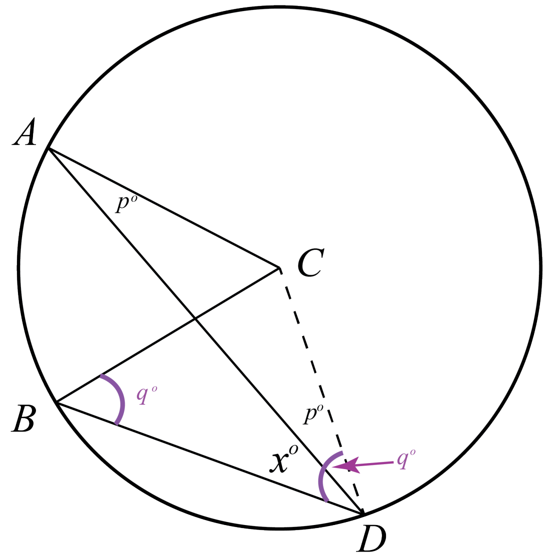


Solution

We construct radius CD. Let $\angle CAD = p^\circ$ and $\angle CBD = q^\circ$.

Since CA and CD are both radii of the circle, $CA = CD$. So $\triangle CAD$ is isosceles and $\angle CDA = \angle CAD = p^\circ$. Since the angles in a triangle add to 180°, $\angle ACD = (180 - 2p)^\circ$.

Since CB and CD are both radii of the circle, $CB = CD$. So $\triangle CBD$ is isosceles and $\angle CDB = \angle CBD = q^\circ$. Since the angles in a triangle add to 180°, $\angle BCD = (180 - 2q)^\circ$.



Now,

$$\begin{aligned}\angle ACB &= \angle ACD - \angle BCD \\ &= (180-2p)^\circ - (180-2q)^\circ \\ &= (2q-2p)^\circ \\ &= 2(q-p)^\circ\end{aligned}$$

Since $\angle CDB = \angle CDA + \angle ADB$, we have $q = p + x$.

Thus, $\angle ACB = 2(q-p)^\circ = 2x^\circ$.

NOTE: In general, the angle inscribed at the centre of a circle is twice the size of the angle inscribed at the circumference by the same chord. That is, the angle inscribed by chord AB at the centre of the circle ($\angle ACB$) is double the angle inscribed by chord AB on the circumference ($\angle ADB$).

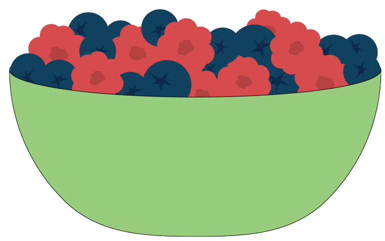
Problem of the Week

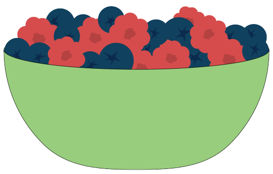
Problem E

Fruit Bowl

A bowl contains raspberries and blueberries. Natacha added 10 more raspberries to the bowl, then determined that $\frac{3}{10}$ of the berries in the bowl were raspberries. Jing then added 20 more blueberries to the bowl, then determined that $\frac{1}{4}$ of the berries in the bowl were raspberries.

What fraction of the original berries in the bowl were raspberries?





Problem of the Week

Problem E and Solution

Fruit Bowl

Problem

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What fraction of the original berries in the bowl were raspberries?

Solution

Let r represent the number of raspberries originally in the bowl, and let b represent the number of blueberries originally in the bowl.

After Natacha added 10 more raspberries, there were $r + 10$ raspberries and b blueberries. Since $\frac{3}{10}$ of the berries in the bowl were raspberries,

$$\begin{aligned} \frac{r+10}{r+10+b} &= \frac{3}{10} \\ 10r + 100 &= 3r + 30 + 3b \\ 7r - 3b + 70 &= 0 \qquad (1) \end{aligned}$$

After Jing added 20 more blueberries, there were $r + 10$ raspberries and $b + 20$ blueberries. Since $\frac{1}{4}$ of the berries in the bowl were raspberries,

$$\begin{aligned} \frac{r+10}{r+10+b+20} &= \frac{1}{4} \\ 4r + 40 &= r + b + 30 \\ 3r + 10 &= b \qquad (2) \end{aligned}$$

Substituting equation (2) into equation (1),

$$\begin{aligned} 7r - 3(3r + 10) + 70 &= 0 \\ 7r - 9r - 30 + 70 &= 0 \\ -2r + 40 &= 0 \\ r &= 20 \end{aligned}$$

Then $b = 3(20) + 10 = 70$. Therefore, there were originally 20 raspberries and 70 blueberries in the bowl. Thus, the fraction of the original berries in the bowl that were raspberries is $\frac{20}{20+70} = \frac{20}{90} = \frac{2}{9}$.

Problem of the Week
Problem E
Nested Squares

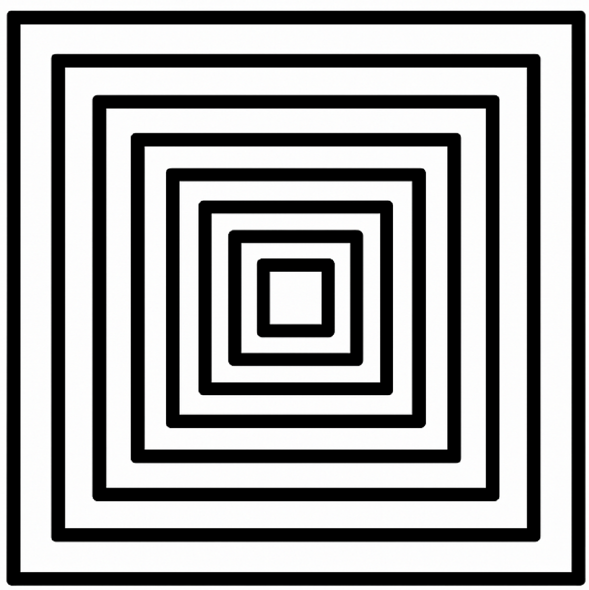
The prime factorization of 20 is $2^2 \times 5$.

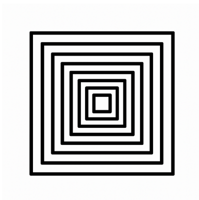
The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 2025^{2025} are perfect squares?





Problem of the Week
Problem E and Solution
Nested Squares

Problem

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Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 2025^{2025} are perfect squares?

Solution

First, let's look at the prime factorization of four different perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \ 129\,600 = 2^6 \times 3^4 \times 5^2$$

Note that, in each case, the exponent on each of the prime factors is even. In fact, a positive integer is a perfect square exactly when the exponent on each of the prime factors in its prime factorization is an even integer greater than or equal to zero. Now

$$\begin{aligned} 2025^{2025} &= (3^4 \times 5^2)^{2025} \\ &= (3^4)^{2025} \times (5^2)^{2025} \\ &= 3^{8100} \times 5^{4050} \end{aligned}$$

All positive divisors of 2025^{2025} will be of the form $3^k \times 5^n$, $0 \leq k \leq 8100$, and $0 \leq n \leq 4050$, where k and n are each integers.

For $3^k \times 5^n$ to be a perfect square, k must be an even integer such that $0 \leq k \leq 8100$ and n must be an even integer such that $0 \leq n \leq 4050$.

There are $8100 \div 2 = 4050$ even integers from 1 to 8100. The number 0 is also even, so there are 4051 possible values of k.

There are 2025 even integers from 1 to 4050. The number 0 is also even, so there are 2026 possible values of n.

For each of the 4051 values of k, there are 2026 values of n so there are $4051 \times 2026 = 8\,207\,326$ perfect square divisors of 2025^{2025}.

Therefore, 2025^{2025} has $8\,207\,326$ positive divisors that are perfect squares. This is over 8 million perfect square divisors!

Problem of the Week

Problem E

Luminous Detail

A *pixel* is the smallest unit of a digital image.

The number of pixels/cm in each of the horizontal and vertical directions of a digital image affects the quality of the image. The more pixels/cm, the sharper the image will be.

A small monitor has dimensions 15 cm by 10 cm and has 80 pixels/cm in each dimension. The total number of pixels is $(15 \times 80) \times (10 \times 80) = 960\,000$.

The manufacturer wants to build a new monitor with $2\,145\,624$ pixels. To accomplish this, both the length and width of the screen will be increased by $n\%$ and the number of pixels/cm in each dimension will be increased by $2n\%$. Determine the dimensions of the new monitor and the new number of pixels/cm.

Problem of the Week

Problem E and Solution

Luminous Detail

Problem

A *pixel* is the smallest unit of a digital image.

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Solution

Let $a > 0$ represent the percentage increase, expressed as a decimal, in each dimension of the monitor. Then $2a$ represents the percentage increase, expressed as a decimal, in the number of pixels/cm. (So $n = 100a$ and $2n = 200a$.)

We know that

[(New Length)×(New pixels/cm)]×[(New Width)×(New pixels/cm)] = Total Number of Pixels

Thus,

$$\begin{aligned}
[15(1+a) \times 80(1+2a)] \times [10(1+a) \times 80(1+2a)] &= 2\,145\,624 \\
(15)(80)(10)(80)(1+a)^2(1+2a)^2 &= 2\,145\,624 \\
960\,000(1+a)^2(1+2a)^2 &= 2\,145\,624 \\
(1+a)^2(1+2a)^2 &= 2.235\,025
\end{aligned}$$

Taking the square root of both sides, we have $(1+a)(1+2a) = \pm 1.495$.

Since $a > 0$, we have $(1+a)(1+2a) > 0$. Thus, $(1+a)(1+2a) = 1.495$.

Expanding and simplifying, we have $2a^2 + 3a + 1 = 1.495$, and so $2a^2 + 3a - 0.495 = 0$.

Using the quadratic formula, $a = \dfrac{-3 \pm \sqrt{9 - 4(2)(-0.495)}}{4} = \dfrac{-3 \pm \sqrt{12.96}}{4} = \dfrac{-3 \pm 3.6}{4}$.

It follows that $a = 0.15$ or $a = -1.65$. Since we are looking for a percentage increase, then we must have $a > 0$ and so $a = -1.65$ is inadmissible.

Therefore, the dimensions of the screen must each be increased by 15% and the numbers of pixels/cm must be increased by $2 \times 15\% = 30\%$.

The increased length is $15 \times 1.15 = 17.25$ cm and the increased width is $10 \times 1.15 = 11.5$ cm. The increased number of pixels/cm is $80 \times 1.3 = 104$ pixels/cm.

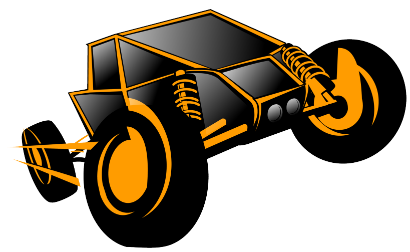
Problem of the Week

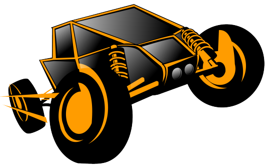
Problem E

Straight Ahead

Albertino and Mara are driving their remote control cars in a large field. Albertino's car starts 10 km north of Mara's car. At the same time, Albertino starts driving his car south and Mara start's driving her car east. Both cars travel at a constant speed of 40 km/hr.

After how many minutes of driving will the **total** distance travelled by the cars be equal to the distance between the cars?





Problem of the Week

Problem E and Solution

Straight Ahead

Problem

Albertino and Mara are driving their remote control cars in a large field. Albertino's car starts 10 km north of Mara's car. At the same time, Albertino starts driving his car south and Mara start's driving her car east. Both cars travel at a constant speed of 40 km/hr.

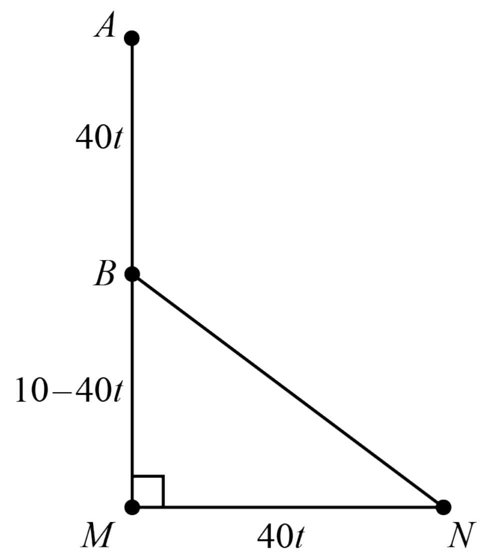
After how many minutes of driving will the **total** distance travelled by the cars be equal to the distance between the cars?

Solution

Let t be the time in hours until the total distance travelled by the cars is equal to the distance between the cars. Since each car is travelling at 40 km/hr, the distance traveled by each car in t hours is $40t$ km. We will now consider three cases: first when $40t < 10$, then when $40t = 10$, and finally when $40t > 10$.

- **Case 1:** $40t < 10$

  In this case, at time t, Albertino's car had not yet reached the initial position of Mara's car. The diagram shows the initial positions of Albertino's car, A, and Mara's car, M, as well as their positions after t hours of driving, which are represented as B and N, respectively. Point B is $40t$ km south of point A, and point N is $40t$ km east of point M. Since the initial distance between the cars, AM, is 10 km, it follows that $BM = (10 - 40t)$ km.



  Since each car travelled $40t$ km in t hours, the total distance travelled by the cars is $2 \times 40t = 80t$ km. Thus, we want to determine the value of t when $BN = 80t$. Since $\triangle BMN$ is a right-angled triangle, we can use the Pythagorean Theorem.

$$\begin{aligned}
BN^2 &= BM^2 + MN^2 \\
(80t)^2 &= (10 - 40t)^2 + (40t)^2 \\
6400t^2 &= 100 - 800t + 1600t^2 + 1600t^2
\end{aligned}$$

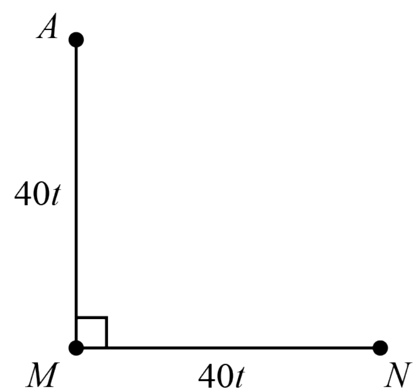
$$\begin{aligned}
3200t^2 + 800t - 100 &= 0 \\
32t^2 + 8t - 1 &= 0
\end{aligned}$$

Using the quadratic formula,

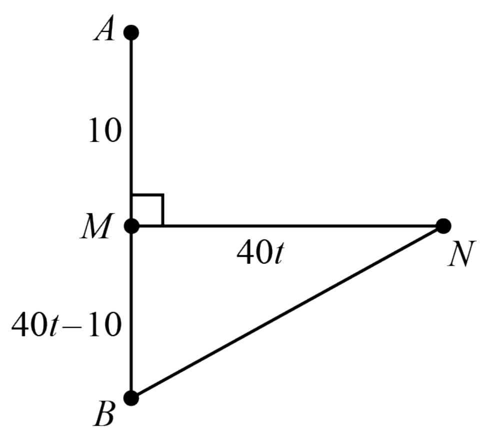
$$t = \frac{-8 \pm \sqrt{8^2 - 4(32)(-1)}}{2(32)} = \frac{-8 \pm \sqrt{192}}{64} = \frac{-1 \pm \sqrt{3}}{8}$$

Since $t > 0$, $t = \frac{-1+\sqrt{3}}{8} \approx 0.09150635$ hours. This is equivalent to approximately 5.49 minutes, or 5 minutes and 29 seconds.

- **Case 2:** $40t = 10$
  In this case, at time t, Albertino's car was at the initial position of Mara's car. So the distance between the cars is the distance that Mara's car travelled, which is $40t$ km. As before, the total distance travelled by the cars is $80t$ km. If these values are equal, then $40t = 80t$. The only solution to this is $t = 0$, but since $t > 0$, this solution is inadmissable. Thus, it is not possible that $40t = 10$.



- **Case 3:** $40t > 10$
  In this case, at time t, Albertino's car had travelled past the initial position of Mara's car. Since the distance between the initial positions of the cars is 10 km, and Albertino's car travelled $40t$ km, it follows that his car travelled $40t - 10$ km past the initial position of Mara's car.

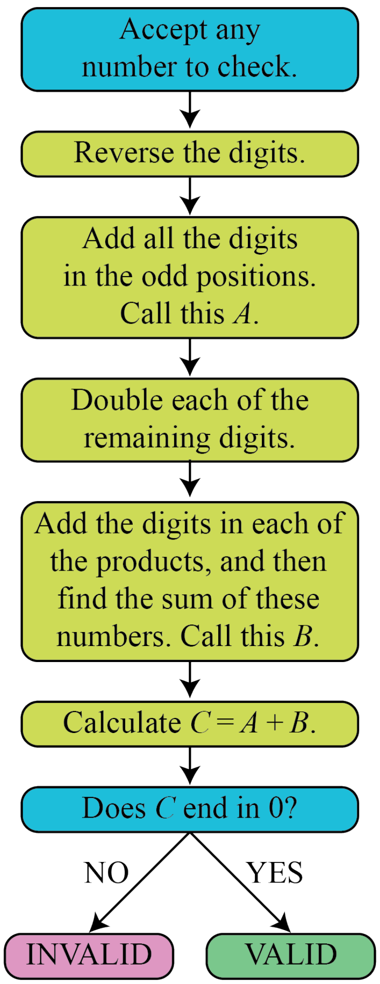


  As before, the total distance travelled by the cars is $80t$ km. If this is equal to the distance between the cars, then $BN = 80t$. Then, in $\triangle BMN$, $BM + MN = 40t - 10 + 40t = 80t - 10 < 80t = BN$. Therefore, this does not satisfy the triangle inequality, so there is no solution for t. Thus, it is not possible that $40t > 10$.

Therefore there is only one solution. After approximately 5.49 minutes, or 5 minutes and 29 seconds, the total distance travelled by the cars will be equal to the distance between the cars.

Problem of the Week
Problem E
Check Please 3

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$

- Calculate B:

$$\begin{aligned} B &= (1 + 8) + (1 + 0) + 2 \\ &= 9 + 1 + 2 \\ &= 12 \end{aligned}$$

- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 2643 $XY5X$ 081 is a valid card number when verified by the Luhn Algorithm, where X and Y are single digits and $X \leq Y$. Determine all possible values of X and Y.

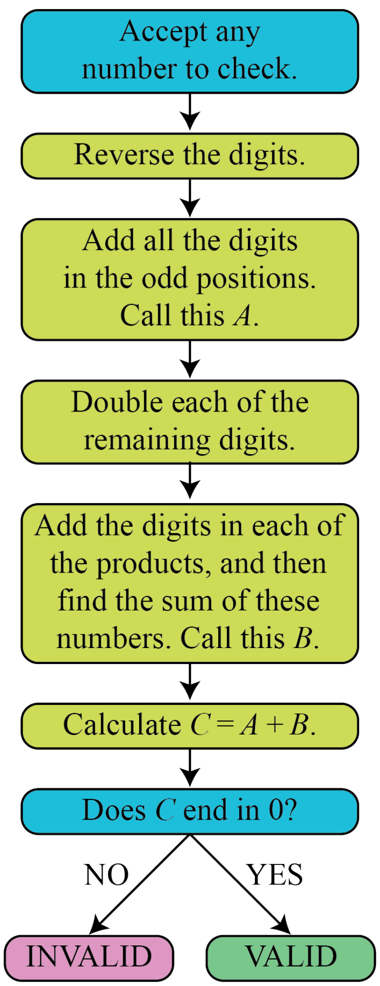
Problem of the Week

Problem E and Solution

Check Please 3

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
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- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$
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- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$
  $$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$
- Double remaining digits:
  $$2 \times 8 = 16$$
  $$2 \times 7 = 14$$
  $$2 \times 3 = 6$$
- Calculate B:
  $$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$
- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 2643 $XY5X$ 081 is a valid card number when verified by the Luhn Algorithm, where X and Y are single digits and $X \leq Y$. Determine all possible values of X and Y.

Solution

When the digits of the number are reversed the resulting number is $180\ X5YX\ 3462$. The sum of the digits in the odd positions is $A = 1 + 0 + 5 + X + 4 + 2 = 12 + X$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$2 \times 8 = 16;\ 2 \times X = 2X;\ 2 \times Y = 2Y;\ 2 \times 3 = 6;\ 2 \times 6 = 12$$

Let m represent the sum of the digits of $2X$ and n represent the sum of the digits of $2Y$. When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + m + n + 6 + (1 + 2) = 16 + m + n$$

C is the sum of A and B, so $C = 12 + X + 16 + m + n = 28 + X + m + n$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what possible sums can be obtained? We summarize these in the following table.

Original Digit (X or Y)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2X$ or $2Y$)	0	2	4	6	8	10	12	14	16	18
Sum of the Digits of $2X$ or $2Y$ (m or n)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of $2X$ or $2Y$ can be any integer from 0 to 9, inclusive. It follows that the only values for m or n are the integers from 0 to 9.

Since the number is valid, $C = 28 + X + m + n$ must end in zero.

What are the possible values to consider for C? Since X, m, and n are each integers from 0 to 9, the minimum value of $X + m + n$ is 0, and the maximum is 27. Thus, the minimum value for C is 28 and the maximum is $28 + 27 = 55$. It follows that the only valid possibilities for C that end in zero are 30, 40, and 50. We will consider each of the three possibilities.

- Case 1: $C = 30$.
  Then $X + m + n = 2$. The only possibility is $X = m = 0$ and $n = 2$. Then $Y = 1$. Thus, one solution is $(X, Y) = (0, 1)$.

- Case 2: $C = 40$.
  Then $X + m + n = 12$. We summarize the possibilities in the following table.

X	m	$n = 12 - X - m$	Y	Valid?
0	0	12	not possible, $n > 9$	no, $n > 9$
1	2	9	9	yes, $X \leq Y$
2	4	6	3	yes, $X \leq Y$
3	6	3	6	yes, $X \leq Y$
4	8	0	0	no, $X > Y$
5	1	6	3	no, $X > Y$
6	3	3	6	yes, $X \leq Y$
7	5	0	0	no, $X > Y$
8	7	-3	not possible, $n < 0$	no, $n < 0$
9	9	-6	not possible, $n < 0$	no, $n < 0$

Therefore, this case produces four valid possibilities for (X, Y): $(1, 9)$, $(2, 3)$, $(3, 6)$ and $(6, 6)$.

- Case 3: $C = 50$.
  Then $X + m + n = 22$. We summarize the possibilities in the following table.

X	m	$n = 22 - X - m$	Y	Valid?
0	0	22	not possible, $n > 9$	no, $n > 9$
1	2	19	not possible, $n > 9$	no, $n > 9$
2	4	16	not possible, $n > 9$	no, $n > 9$
3	6	13	not possible, $n > 9$	no, $n > 9$
4	8	10	not possible, $n > 9$	no, $n > 9$
5	1	16	not possible, $n > 9$	no, $n > 9$
6	3	13	not possible, $n > 9$	no, $n > 9$
7	5	10	not possible, $n > 9$	no, $n > 9$
8	7	7	8	yes, $X \leq Y$
9	9	4	2	no, $X > Y$

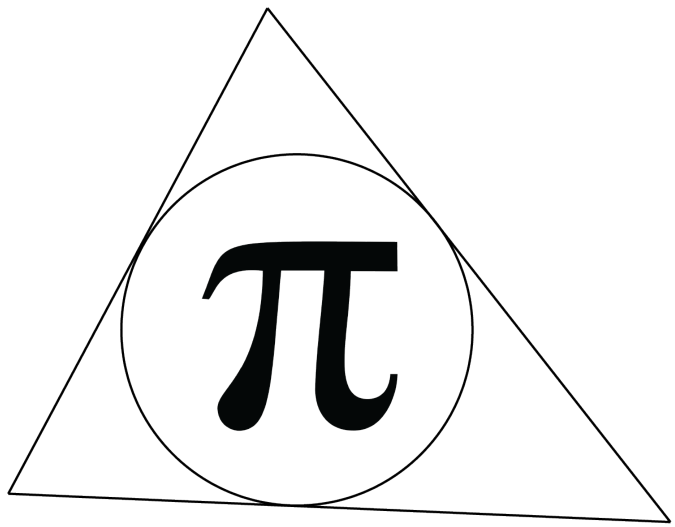
Therefore, this case produces one valid possibility for (X, Y): $(8, 8)$.

We have examined all possible values for C. Therefore, there are six valid possibilities for (X, Y): $(0, 1)$, $(1, 9)$, $(2, 3)$, $(3, 6)$, $(6, 6)$, and $(8, 8)$.

These correspond to the following six card numbers, which are indeed valid by the Luhn Algorithm:
2643 0150 081, 2643 1951 081, 2643 2352 081, 2643 3653 081, 2643 6656 081, and 2643 8858 081.

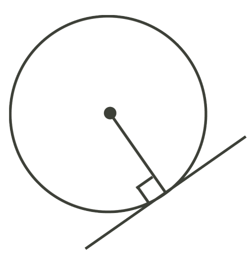
Problem of the Week
Problem E
Pizza Pi Box

A box in the shape of a triangle with side lengths 13, 14, and 15 is being used to pack a pizza pi. The pizza is in the shape of a circle and just touches each of the three sides of the triangular box. What is the area of the circle?

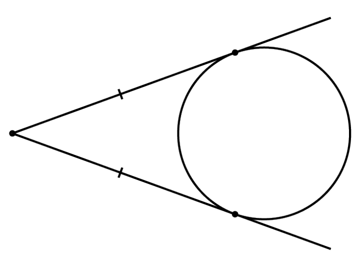


NOTE: For this problem, the following known results about circles may be useful:

- If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



- If two tangent segments are drawn to a circle from a point outside of the circle, then the lengths of those segments are equal.



Problem of the Week
Problem E and Solution
Pizza Pi Box

Problem

A box in the shape of a triangle with side lengths 13, 14, and 15 is being used to pack a pizza pi. The pizza is in the shape of a circle and just touches each of the three sides of the triangular box. What is the area of the circle?

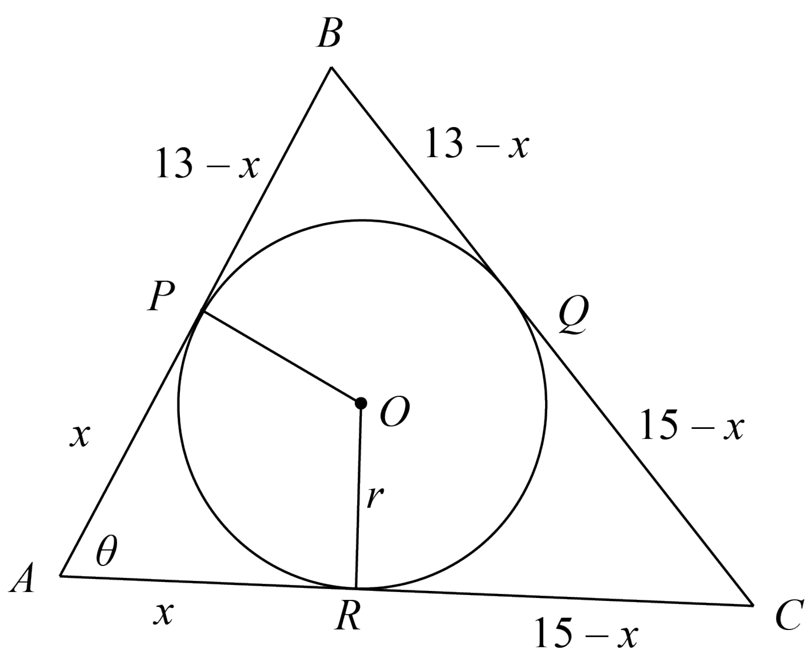
Solution

We label the triangle ABC, with $AB = 13$, $BC = 14$, and $AC = 15$.

Let P be the point where the circle touches the triangle on AB, Q be the point where the circle touches the triangle on BC, and R be the point where the circle touches the triangle on AC.

Let $\angle BAC = \theta$, and let the circle be centred at point O with radius r.

Using the second given result about circles, we know that $AP = AR$, $CR = CQ$, and $BP = BQ$. Let $AP = AR = x$. Then $BP = AB - AP = 13 - x$. Since $BP = BQ$, we have $BQ = BP = 13 - x$. Also, $CR = AC - AR = 15 - x$. Since $CR = CQ$, we have $CQ = 15 - x$.



Since $BC = 14$ and $BC = BQ + CQ$, we have $13 - x + 15 - x = 14$, thus $28 - 2x = 14$, and so $x = 7$.

Using the cosine law in $\triangle ABC$,

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\ 14^2 &= 13^2 + 15^2 - 2(13)(15)\cos(\theta) \\ \cos\theta &= \frac{14^2 - 13^2 - 15^2}{-2(13)(15)} \\ &= \frac{-198}{-390} \\ &= \frac{33}{65} \end{aligned}$$

Consider $\triangle AOR$ and $\triangle AOP$. Since $AR = AP$, $OR = OP = r$, and $AO = AO$ (same length), $\triangle AOR$ is congruent to $\triangle AOP$. Thus, $\angle OAP = \angle OAR = \frac{\theta}{2}$. Further, using the first given result about circles, we know that $\triangle AOR$ and $\triangle AOP$ are right-angled.

In $\triangle AOR$, $\tan\left(\frac{\theta}{2}\right) = \frac{r}{x} = \frac{r}{7}$. Thus, $r = 7\tan\left(\frac{\theta}{2}\right)$.

Using the identities $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, we obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2} = \frac{49}{65}$, and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} = \frac{16}{65}$.

Since $0 < \frac{\theta}{2} < 90^\circ$, $\cos\left(\frac{\theta}{2}\right) > 0$ and $\sin\left(\frac{\theta}{2}\right) > 0$. Thus, $\cos\left(\frac{\theta}{2}\right) = \frac{7}{\sqrt{65}}$ and $\sin\left(\frac{\theta}{2}\right) = \frac{4}{\sqrt{65}}$.

Therefore, we have

$$\begin{aligned} r &= 7\tan\left(\frac{\theta}{2}\right) \\ &= \frac{7\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \\ &= \frac{7\left(\frac{4}{\sqrt{65}}\right)}{\frac{7}{\sqrt{65}}} \\ &= 4 \end{aligned}$$

Therefore, the area of the circle is $\pi(4)^2 = 16\pi$.

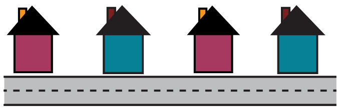
Problem of the Week

Problem E

A Street Called Square

Alyssa, Bilal, Constantine, and Daiyu each live in a house that is numbered with a positive integer.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. Constantine's house number is 31 more than Alyssa's house number, and Daiyu's house number is 31 more than Bilal's house number. What is each person's house number?



Problem of the Week
Problem E and Solution
A Street Called Square

Problem

Alyssa, Bilal, Constantine, and Daiyu each live in a house that is numbered with a positive integer.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. Constantine's house number is 31 more than Alyssa's house number, and Daiyu's house number is 31 more than Bilal's house number. What is each person's house number?

Solution

Both Alyssa's house number and Bilal's house number must be two-digit integers. If Alyssa's house number is a one-digit integer, Bilal's house number would have to be a three-digit integer to create the four-digit perfect square. This means that Constantine's house number would be a two-digit integer and Daiyu's house number would be at least a three-digit integer. Writing Constantine's house number followed by Daiyu's house number would create a five-digit integer. A similar argument can be presented if Bilal's house number is a one-digit integer. Therefore, both Alyssa and Bilal have house numbers that are two-digit integers.

Let Alyssa's house number be x and Bilal's house number be y. Then $100x + y$ is the four-digit integer created by writing Alyssa's house number followed by Bilal's house number. It is given that $100x + y$ is a perfect square, so $100x + y = k^2$ for some positive integer k.

Therefore, Constantine's house number is $(x + 31)$ and Daiyu's house number is $(y + 31)$. The number created by writing Constantine's house number followed by Daiyu's house number is $100(x + 31) + (y + 31)$. This four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer m. Simplifying, we have

$$\begin{aligned} 100x + 3100 + y + 31 &= m^2 \\ 100x + y + 3131 &= m^2 \end{aligned}$$

Substituting $100x + y = k^2$, we obtain $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$.

Since $m^2 - k^2$ is a difference of squares, we have $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m+k$ is a positive integer and $m+k > m-k$. Also, $m-k$ must be a positive integer since $(m+k)(m-k) = 3131$. So we are looking for two positive integers that multiply to 3131. There are two possibilities, 3131×1 or 101×31.

First we will examine $(m+k)(m-k) = 3131 \times 1$. From this we obtain two equations in two unknowns, namely $m+k = 3131$ and $m-k = 1$. Subtracting the second equation from the first gives $2k = 3130$ or $k = 1565$. Then $k^2 = 1565^2 = 2\,449\,225$. This is not a four-digit number, so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m+k)(m-k) = 101 \times 31$. This leads to $m+k = 101$ and $m-k = 31$. Subtracting the second equation from the first, we get $2k = 70$ or $k = 35$. Then $100x + y = k^2 = 1225$. Therefore, $x = 12$ and $y = 25$, since 1225 is the four-digit number formed by writing Alyssa's house number, x, followed by Bilal's house number, y.

Now, Constantine's house number is $x + 31 = 12 + 31 = 43$ and Daiyu's house number is $y + 31 = 25 + 31 = 56$. Notice that $4356 = 66^2$, so it is a four-digit perfect square.

Therefore, Alyssa's house number is 12, Bilal's house number is 25, Constantine's house number is 43, and Daiyu's house number is 56.

Problem of the Week

Problem E

Stock the Shelves

A local food bank has created a unique 100-day plan for collecting canned food donations.

Day 1 Goal: Collect 50 cans of food.

Day 2 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 1.

Day 3 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 2.

Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

$\vdots$

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100^{th} day?

NOTE:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Problem of the Week
Problem E and Solution
Stock the Shelves

Problem
A local food bank has created a unique 100-day plan for collecting canned food donations.

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$\vdots$

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100^{th} day?

Solution
We will introduce function notation to represent the information in the problem.

Let n represent the day number and $f(n)$ represent the number of cans collected on day n.

We know that on the first day, 50 cans were collected. So, $f(1) = 50$.

On the second day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 1. So, $f(2) = 3 + 2 + f(1) = 3 + 2 + 50 = 55$.

On the third day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 2. So, $f(3) = 3 + 3 + f(2) = 3 + 3 + 55 = 61$.

On the fourth day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 3. So, $f(4) = 3 + 4 + f(3) = 3 + 4 + 61 = 68$.

On the n^{th} day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day $(n - 1)$. So, $f(n) = 3 + n + f(n - 1)$.

This sequence of cans collected can be defined recursively as follows:

$$f(n) = \begin{cases} 50, & \text{if } n = 1 \\ f(n-1) + n + 3, & \text{if } n \geq 2 \text{ and } n \in \mathbb{Z} \end{cases}$$

We want to find $f(100)$, the 100^{th} term in the sequence.

Solution 1

If $f(n-1)$ and $f(n)$, with $n \geq 2$, are adjacent terms in the sequence of cans collected, then $f(n) = f(n-1) + n + 3$. Rearranging, $f(n) - f(n-1) = n + 3$. That is, the difference between consecutive terms will be the term number of the term in the higher position plus 3.

Thus, $f(100)$ will be equal to $f(1) = 50$, plus the sum of the term numbers from 2 to 100, plus 3 a total of 99 times. That is,

$$\begin{aligned} f(100) &= 50 + (2 + 3 + 4 + \cdots + 100) + 3(99) \\ &= 50 + (1 + 2 + 3 + \cdots + 100) - 1 + 3(99) \\ &= 50 + \frac{100(101)}{2} - 1 + 3(99) \\ &= 50 + 5050 - 1 + 297 \\ &= 5396 \end{aligned}$$

Therefore, 5396 cans are collected on the 100^{th} day.

Solution 2

We summarize the number of cans collected on the first 6 days in a table:

Day Number	1	2	3	4	5	6
Number of Cans Collected	50	55	61	68	76	85

If we treat the day number as the independent variable, say x, and the number of cans collected as the dependent variable, say y, we can make some conclusions. First, Δx is constant. However, Δy, the first differences, are not constant. Therefore, the function is not linear.

The first differences of the given values are 5, 6, 7, 8, and 9. If we calculate the second differences, each is 1. That is, for a constant Δx, the first differences are not constant but the second differences are constant. We will leave it to the reader to justify that this pattern is true for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $x \geq 1, x \in \mathbb{Z}$.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 50$, so

$$a + b + c = 50 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 55$, so

$$4a + 2b + c = 55 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 61$, so

$$9a + 3b + c = 61 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 5$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 6$, which we call equation (5).

We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 5$, and so $b = 5 - \frac{3}{2} = \frac{7}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{7}{2}$ into equation (1), we have $\frac{1}{2} + \frac{7}{2} + c = 50$, and so $c = 50 - 4 = 46$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{7}{2}x + 46$, where $x \geq 1, x \in \mathbb{Z}$.

To determine the value collected on the 100^{th} day, we evaluate $f(x)$ at $x = 100$.

$$f(100) = \frac{1}{2}(100)^2 + \frac{7}{2}(100) + 46 = 5000 + 350 + 46 = 5396$$

Therefore, 5396 cans are collected on the 100^{th} day.

Using this approach, we can find the amount collected on any day, since we have a general formula for the value collected given the day number. That is, the amount collected on day n is $f(n) = \frac{1}{2}n^2 + \frac{7}{2}n + 46$, where $n \geq 1, n \in \mathbb{Z}$.

Solution 3

When $n \geq 2$, we have $f(n) = f(n-1) + n + 3$. Rearranging, we obtain $f(n) - f(n-1) = n + 3$. Using $f(n) - f(n-1) = n + 3$ with integer values of n from 100 to 2, we have

$$\begin{aligned}
f(100) - f(99) &= 100 + 3 \\
f(99) - f(98) &= 99 + 3 \\
f(98) - f(97) &= 98 + 3 \\
&\vdots \\
f(3) - f(2) &= 3 + 3 \\
f(2) - f(1) &= 2 + 3
\end{aligned}$$

If we add all of the terms on the left side of each equal sign and then simplify, we are left with only $f(100) - f(1)$. If we add all of the terms on the right side of each equal sign, we get

$$2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3)$$

Therefore,

$$\begin{aligned}
f(100) - f(1) &= 2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3) \\
&= (1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100) - 1 + 99(3) \\
&= \frac{100(101)}{2} - 1 + 99(3) \\
&= 5050 - 1 + 297 \\
&= 5346
\end{aligned}$$

Since $f(1) = 50$, we have $f(100) - 50 = 5346$, and so $f(100) = 5396$.

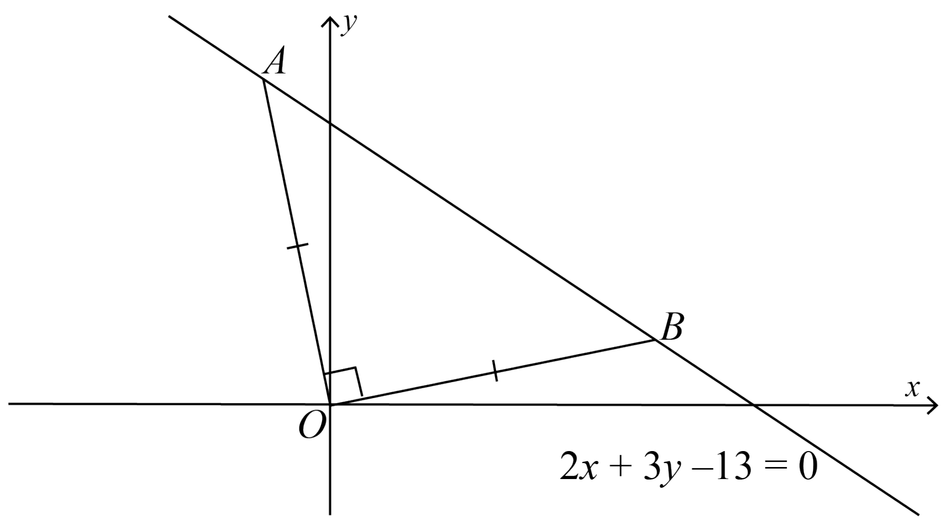
That is, 5396 cans are collected on the 100^{th} day.

Extension: Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?

Problem of the Week
Problem E
Tilted Triangle

Suppose that $\triangle OAB$ is such that vertex O is located at the origin, and vertices A and B lie on the line $2x + 3y - 13 = 0$ with $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.

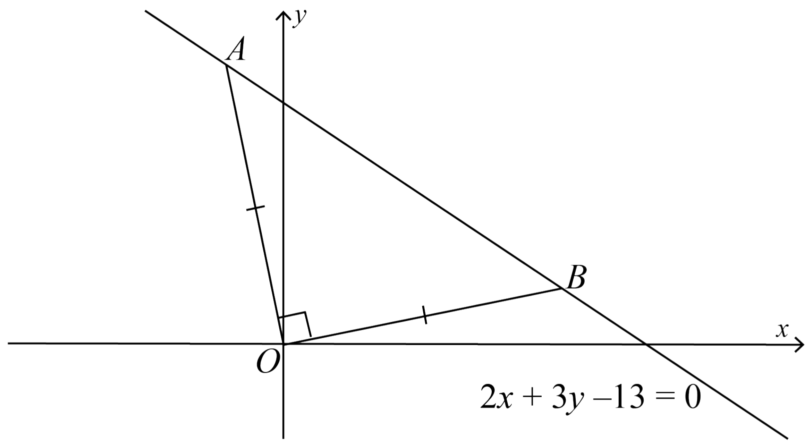


Problem of the Week
Problem E and Solution
Tilted Triangle

Problem

Suppose that $\triangle OAB$ is such that vertex O is located at the origin, and vertices A and B lie on the line $2x + 3y - 13 = 0$ with $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.



Solution

By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$.

Since points A and B lie on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$.

Since $OA = OB$, we have

$$\begin{aligned}
OA^2 &= OB^2\\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2\\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9}\\
9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169\\
13a^2 - 52a + 169 &= 13b^2 - 52b + 169\\
13a^2 - 13b^2 - 52a + 52b &= 0\\
a^2 - b^2 - 4a + 4b &= 0
\end{aligned}$$

By factoring, we then obtain $(a+b)(a-b) - 4(a-b) = 0$ or $(a-b)(a+b-4) = 0$.

This implies that, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$.

From here we proceed with two different solutions.

Solution 1

Since B has coordinates $\left(b, \frac{-2b+13}{3}\right)$, the slope of OB is equal to $\dfrac{\frac{-2b+13}{3}}{b} = \dfrac{-2b+13}{3b}$.

Since A has coordinates $\left(a, \frac{-2a+13}{3}\right)$, the slope of OA is equal to $\dfrac{\frac{-2a+13}{3}}{a} = \dfrac{-2a+13}{3a}$.

Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of OA is the negative reciprocal of the slope of OB. Therefore,

$$\begin{aligned}\frac{-2a+13}{3a} &= -\frac{1}{\frac{-2b+13}{3b}} \\ &= \frac{-3b}{-2b+13}\end{aligned}$$

Simplifying, we obtain

$$\begin{aligned}(-2a+13)(-2b+13) &= (-3b)(3a) \\ 4ab - 26a - 26b + 169 &= -9ab \\ 13ab - 26a - 26b + 169 &= 0 \\ ab - 2a - 2b + 13 &= 0\end{aligned}$$

Substituting $a = 4 - b$, we have

$$\begin{aligned}(4-b)b - 2(4-b) - 2b + 13 &= 0 \\ 4b - b^2 - 8 + 2b - 2b + 13 &= 0 \\ b^2 - 4b - 5 &= 0\end{aligned}$$

Factoring, this becomes $(b-5)(b+1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$.

In each case, the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

Since $\triangle AOB$ is a right-angled triangle, we can use OB as the base and OA as the height in the formula for the area of a triangle.

Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26} \times \sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Solution 2

Since $a = 4 - b$, we can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is right-angled at O, by the Pythagorean Theorem, $AB^2 = OA^2 + OB^2$. Since $OA = OB$, this can be written $AB^2 = 2OB^2$. Thus,

$$\begin{aligned}(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left(b^2 + \left(\frac{-2b+13}{3}\right)^2\right)\\ (2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left(b^2 + \frac{4b^2-52b+169}{9}\right)\\ 4b^2 - 16b + 16 + \frac{16b^2-64b+64}{9} &= 2b^2 + \frac{8b^2-104b+338}{9}\end{aligned}$$

Multiplying by 9 and simplifying, we obtain

$$\begin{aligned}36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338\\ 52b^2 - 208b + 208 &= 26b^2 - 104b + 338\\ 26b^2 - 104b - 130 &= 0\\ b^2 - 4b - 5 &= 0\end{aligned}$$

Factoring, this becomes $(b-5)(b+1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

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In each case, the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

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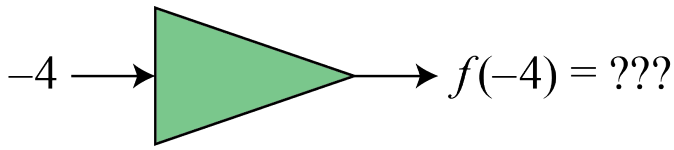
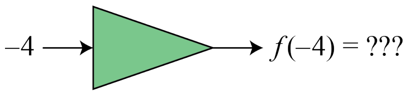
Problem of the Week

Problem E

How Does it Function?

The function $f(x) = x^5 - 4x^4 + ax^3 - x^2 + bx - 8$ has a value of 128 when $x = 4$. Determine the value of the function when $x = -4$.





Problem of the Week

Problem E and Solution

How Does it Function?

Problem

The function $f(x) = x^5 - 4x^4 + ax^3 - x^2 + bx - 8$ has a value of 128 when $x = 4$. Determine the value of the function when $x = -4$.

Solution

We know that the function has a value of 128 when $x = 4$. That is, $f(4) = 128$. Then,

$$\begin{aligned} f(4) &= 128 \\ (4)^5 - 4(4)^4 + a(4)^3 - (4)^2 + b(4) - 8 &= 128 \\ 1024 - 1024 + 64a - 16 + 4b - 8 &= 128 \\ 64a + 4b &= 152 \end{aligned}$$

We want to determine the value of the function when $x = -4$. That is, we want to determine $f(-4)$. Then,

$$\begin{aligned} f(-4) &= (-4)^5 - 4(-4)^4 + a(-4)^3 - (-4)^2 + b(-4) - 8 \\ &= -1024 - 1024 - 64a - 16 - 4b - 8 \\ &= -64a - 4b - 2072 \end{aligned}$$

However, we know from earlier that $64a + 4b = 152$. Then,

$$f(-4) = -(64a + 4b) - 2072 = -152 - 2072 = -2224$$

Therefore, the value of the function is -2224 when $x = -4$.

Note that we are not given enough information to determine the values of a and b, however enough information is given to determine the value of $f(-4)$.

Computational Thinking (C)

Take me to the cover

Problem of the Week

Problem E

Jumbled Numbers 2

Imani wrote a program that switches digits in numbers. Users enter a 4-digit number and then use red R, blue B, and yellow Y buttons to switch digits in the number as follows:

Button	Function
R	switches the thousands and tens digits
B	switches the hundreds and units (ones) digits
Y	switches the thousands and units (ones) digits

Imani determines all possible results when the number 1234 is entered and a sequence of 0, 1, 2, 3, or 4 buttons is pressed. She records these results in a list.

How many different results, *not* on Imani's list, can be obtained when the number 1234 is entered and a sequence of exactly 5 buttons is pressed?

R Y B

Problem of the Week

Problem E and Solution

Jumbled Numbers 2

Problem

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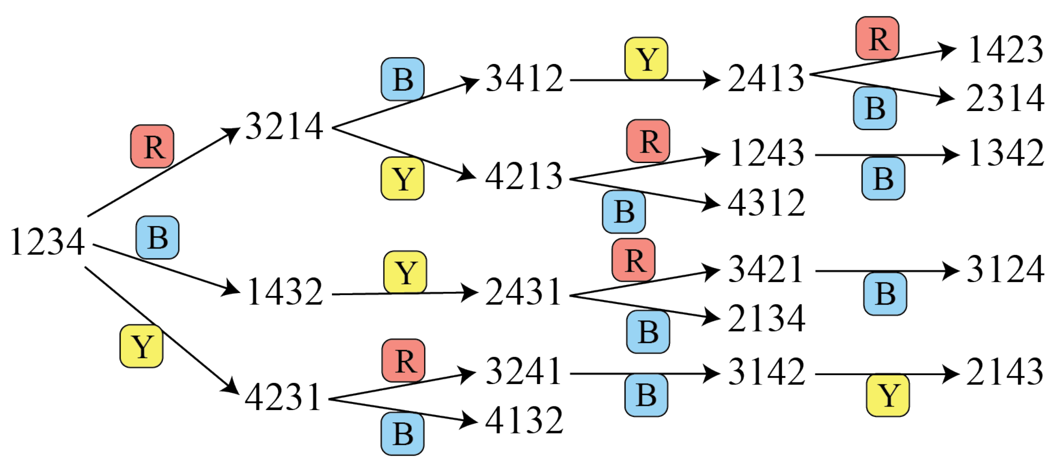
Imani determines all possible results when the number 1234 is entered and a sequence of 0, 1, 2, 3, or 4 buttons is pressed. She records these results in a list.

How many different results, *not* on Imani's list, can be obtained when the number 1234 is entered and a sequence of exactly 5 buttons is pressed?

Solution

We start by making a tree diagram to determine all the numbers in Imani's list. We note that it's possible to obtain the same result using a different sequence of buttons, for example, pressing R B gives the same result as pressing B R. We will create our tree diagram one button at a time, moving top to bottom, and will not include any duplicate results in our tree. This will ensure that the tree shows only the shortest path to reach each of the possible results.



Thus, Imani's list contains 20 numbers. The total number of possible arrangements of four distinct digits is $4 \times 3 \times 2 \times 1 = 24$. Thus, there are $24 - 20 = 4$ possible arrangements that are not on Imani's list. We can determine that these numbers are 1324, 2341, 4123, and 4321. Next we determine how many more buttons are required to obtain each of these.

- From 1423, pressing B gives 1324. Thus, 1324 can be obtained using a sequence of exactly 5 buttons.
- From 2143, pressing B gives 2341. Thus, 2341 can be obtained using a sequence of exactly 5 buttons.
- From 3124, pressing Y gives 4123. Thus, 4123 can be obtained using a sequence of exactly 5 buttons.
- We can obtain 4321 in the following ways:
  - From 1324, pressing Y gives 4321.
  - From 2341, pressing R gives 4321.
  - From 4123, pressing B gives 4321.

  However, since 1324, 2341, and 4123 each require 5 buttons to obtain, it follows that 4321 *cannot* be obtained using a sequence of exactly 5 buttons.

Thus only 3 numbers, namely 1324, 2341, and 4123, are not on Imani's list but can be obtained when the number 1234 is entered and a sequence of exactly 5 buttons is pressed.

Problem of the Week

Problem E

The Third Escape

Ingrid, Jayant, Neela, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Lost Shoes, Memories, Dollhouse, or The Old Office. Each of the rooms had a different-coloured door: black, orange, purple, or yellow. The four groups all started at the same time, but finished at different times.

Use the clues below to match each group with the name of their escape room, the colour of its door, and the order in which they finished.

(1) The group that completed Memories finished before Ingrid's group, but after the group that went through the black door.

(2) Neela's group finished immediately after the group that completed Lost Shoes.

(3) The group that went through the yellow door finished immediately before the group that went through the purple door.

(4) The group that completed Dollhouse went through the orange door and did not finish last.

(5) Radko's group did not finish third and did not go through the orange door.

(6) Jayant's group did not go through the yellow door.

Problem of the Week
Problem E and Solution
The Third Escape

Problem

Ingrid, Jayant, Neela, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Lost Shoes, Memories, Dollhouse, or The Old Office. Each of the rooms had a different-coloured door: black, orange, purple, or yellow. The four groups all started at the same time, but finished at different times.

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(2) Neela's group finished immediately after the group that completed Lost Shoes.

(3) The group that went through the yellow door finished immediately before the group that went through the purple door.

(4) The group that completed Dollhouse went through the orange door and did not finish last.

(5) Radko's group did not finish third and did not go through the orange door.

(6) Jayant's group did not go through the yellow door.

Solution

First we will look at the door colours. From clue (1) we learn about three groups. Of these groups, the group that went through the black door finished before the other two. This tells us that the group that went through the black door finished either first or second. Clue (3) tells us that the group that went through the yellow door finished immediately before the group that went through the purple door. Clue (4) tells us that the group that went through the orange door did not finish last. Putting these together gives the following two possibilities:

1. The group that went through the black door finished first. Then the group that went through the orange door must have finished second, followed by the group that went through the yellow door, and finally the group that went through the purple door.

2. The group that went through the black door finished second. Then the group that went through the orange door must have finished first, the group that went through the yellow door finished third, followed by the group that went through the purple door.

From clue (4), the group that went through the orange door completed Dollhouse. Then from clue (1), the group that went through the black door finished before the group that completed Memories, which finished before Ingrid's group. For each possibility there is only one way to place these. We summarize what we know so far for each of these possibilities in the following tables.

Possibility 1:

	Last	**Third**	**Second**	**First**
Group Leader	Ingrid			
Room Name		Memories	Dollhouse	
Door Colour	purple	yellow	orange	black

Possibility 2:

	Last	**Third**	**Second**	**First**
Group Leader	Ingrid			
Room Name		Memories		Dollhouse
Door Colour	purple	yellow	black	orange

From clue (5), since Radko's group did not finish third and did not go through the orange door, we can conclude Radko's group went through the black door. From clue (6), since Jayant's group did not go through the yellow door, we can conclude Jayant's group went through the orange door. It follows that Neela's group went through the yellow door. We summarize what we know so far for each of these possibilities in the following tables.

Possibility 1:

	Last	**Third**	**Second**	**First**
Group Leader	Ingrid	Neela	Jayant	Radko
Room Name		Memories	Dollhouse	
Door Colour	purple	yellow	orange	black

Possibility 2:

	Last	**Third**	**Second**	**First**
Group Leader	Ingrid	Neela	Radko	Jayant
Room Name		Memories		Dollhouse
Door Colour	purple	yellow	black	orange

Finally, from clue (2) we know Neela's group finished immediately after the group that completed Lost Shoes. We can now eliminate possibility 1. It follows that the group that completed Lost Shoes finished second. Then the group that completed The Old Office finished last. The completed table is shown.

	Last	**Third**	**Second**	**First**
Group Leader	Ingrid	Neela	Radko	Jayant
Room Name	The Old Office	Memories	Lost Shoes	Dollhouse
Door Colour	purple	yellow	black	orange

Data Management (D)

Take me to the cover

Problem of the Week
Problem E
Math Test Math 3

Theo has had three tests in his math class so far. He does the following calculations using the average (mean) of some of his marks.

- First he calculates the average of his first and second test marks. Then he calculates the average of this value with his third test mark to get 74%.
- First he calculates the average of his second and third test marks. Then he calculates the average of this value with his first test mark to get 73.5%.
- First he calculates the average of his first and third test marks. Then he calculates the average of this value with his second test mark to get 76%.

Determine the average his three test marks.

Problem of the Week
Problem E and Solution
Math Test Math 3

Problem

Theo has had three tests in his math class so far. He does the following calculations using the average (mean) of some of his marks.

- First he calculates the average of his first and second test marks. Then he calculates the average of this value with his third test mark to get 74%.
- First he calculates the average of his second and third test marks. Then he calculates the average of this value with his first test mark to get 73.5%.
- First he calculates the average of his first and third test marks. Then he calculates the average of this value with his second test mark to get 76%.

Determine the average his three test marks.

Solution

Let a, b, and c represent the test marks, in order. We are looking for the average (mean) of a, b, and c, which is $\frac{a+b+c}{3}$.

The average of his first and second test marks is $\frac{a+b}{2}$. The average of this value with his third test mark is $\frac{\frac{a+b}{2}+c}{2}$. We know this is equal to 74. Thus,

$$\begin{aligned}\frac{\frac{a+b}{2}+c}{2} &= 74\\ \frac{a+b}{2}+c &= 148\\ a+b+2c &= 296 \qquad (1)\end{aligned}$$

Similarly, we can write the following two equations:

$$\begin{aligned}\frac{\frac{b+c}{2}+a}{2} &= 73.5\\ \frac{b+c}{2}+a &= 147\\ b+c+2a &= 294 \qquad (2)\end{aligned}$$

$$\begin{aligned}\frac{\frac{a+c}{2}+b}{2} &= 76\\ \frac{a+c}{2}+b &= 152\\ a+c+2b &= 304 \qquad (3)\end{aligned}$$

Next we add equations (1), (2), and (3):

$$\begin{aligned}(a+b+2c)+(b+c+2a)+(a+c+2b) &= 296+294+304\\ 4a+4b+4c &= 894\\ a+b+c &= 223.5\\ \frac{a+b+c}{3} &\approx 74.5\end{aligned}$$

Therefore, the average of his three test marks is 74.5%.

Extension: You may have noticed that the average of computed values 74, 73.5, and 76 is also equal to 74.5, the average of the test marks. In general, this will be the case for any three test results, when computed this way. Can you convince yourself of this?

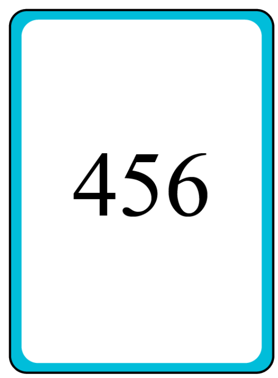
Problem of the Week

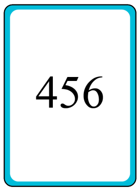
Problem E

A New Deck of Cards

Joanna has a deck of cards. Each card in the deck has a three-digit positive integer on it, and there is exactly one card in the deck for every three-digit positive integer.

Joanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.





Problem of the Week
Problem E and Solution
A New Deck of Cards

Problem

Joanna has a deck of cards. Each card in the deck has a three-digit positive integer on it, and there is exactly one card in the deck for every three-digit positive integer.

Joanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.

Solution

To begin, we determine the number of cards in the deck. There are 999 positive integers less than 1000. Of these, 90 are two-digit integers and 9 are single-digit integers. Therefore, there are $999 - 90 - 9 = 900$ three-digit positive integers, and so 900 cards in the deck.

Next, we determine the digit combinations that sum to 15.

- **Case 1:** One of the digits on the card is a 0.
  Then the other two digits on the card must add to 15. There are two possibilities for the digits. The three digits must be $0, 6, 9$ or $0, 7, 8$.

- **Case 2:** One of the digits on the card is a 1, but the number does not contain a 0.
  Then the other two digits on the card must add to 14. There are three possibilities for the digits. The digits must be $1, 5, 9$ or $1, 6, 8$ or $1, 7, 7$.

- **Case 3:** One of the digits on the card is a 2, but the number does not contain a 0 or 1.
  Then the other two digits on the card must add to 13. There are three possibilities for the digits. The digits must be $2, 4, 9$ or $2, 5, 8$ or $2, 6, 7$.

- **Case 4:** One of the digits on the card is a 3, but the number does not contain a 0, 1, or 2.
  Then the other two digits on the card must add to 12. There are four possibilities for the digits. The digits must be $3, 3, 9$ or $3, 4, 8$ or $3, 5, 7$ or $3, 6, 6$.

- **Case 5:** One of the digits on the card is a 4, but the number does not contain a 0, 1, 2, or 3.
  Then the other two digits on the card must add to 11. There are two possibilities for the digits. The digits must be $4, 4, 7$ or $4, 5, 6$.

- **Case 6:** One of the digits on the card is a 5, but the number does not contain a 0, 1, 2, 3, or 4.
  Then the other two digits on the card must add to 10. There is only one possibility for the digits. The digits must be $5, 5, 5$.

Now that we know what combinations of digits can be on the cards, we can determine the number of cards that can be created from each combination.

- **Case (a):** One of the digits on the card is 0.
  Earlier we found that there were two such digit combinations: $0, 6, 9$ and $0, 7, 8$. This is a special case since 0 cannot appear in the number as the hundreds digit for the number to be a three-digit number. For each of the two digit combinations, the 0 can be placed in two ways, in the tens digit or the units digit. Once the 0 is placed, the other two numbers can be placed in the remaining two spots in two ways. Thus, each digit combination can form $2 \times 2 = 4$ three-digit numbers. Since there are two digit combinations, there are $2 \times 4 = 8$ cards in the deck that contain a 0 whose digits add to 15.

- **Case (b):** All three digits on the card are different and the number does not contain a 0.
  From the earlier cases, there are eight such combinations: $1, 5, 9$, and $1, 6, 8$, and $2, 4, 9$, and $2, 5, 8$, and $2, 6, 7$, and $3, 4, 8$, and $3, 5, 7$, and $4, 5, 6$. For each of these combinations, the hundreds digit can be placed in three ways. For each of these three choices for hundreds digit, the tens digit can be placed in two ways. Once the hundreds digit and tens digit are selected, the units digit must get the third number. So each combination can form $3 \times 2 = 6$ different numbers. Since there are eight digit combinations, there are $8 \times 6 = 48$ cards in the deck that contain three different digits other than 0 whose digits add to 15.

- **Case (c):** Two of the digits on the card are the same and the number does not contain a 0.
  From the earlier cases, there are four such combinations: $1, 7, 7$, and $3, 3, 9$, and $3, 6, 6$, and $4, 4, 7$. For each of these combinations, the unique number can be placed in one of three spots. Once the unique number is placed the other two numbers must go in the remaining two spots. So each digit combination can form three different numbers. Since there are four digit combinations, there are $4 \times 3 = 12$ cards in the deck that do not contain a 0 but contain two digits the same and whose digits add to 15.

- **Case (d):** The three digits on the card are the same.
  From the earlier cases we discovered only one such combination: $5, 5, 5$. Only one card can be produced using the numbers from this combination.

Combining the counts from the above four cases, there are $8 + 48 + 12 + 1 = 69$ cards in the deck with a digit sum of 15. Therefore, the probability that Joanna selects card whose digits add to 15 is $\frac{69}{900} = \frac{23}{300}$. This translates to approximately a 7.7% chance.

A game is considered *fair* if there is close to a 50% chance of winning. If Joanna was playing a game where she can win by drawing a card whose digits sum to 15, then this game is definitely not fair.

Can you create a game using this specific deck of cards that is reasonably fair and fun to play?

Problem of the Week
Problem E
Favourite Number 3

Matej told his friend Clare the following properties about his favourite number:

- It is a positive seven-digit integer.
- It contains each of the digits from 1 through 7 exactly once.
- The first digit is not 1.
- The second digit is not 2.

Clare then wrote down a number satisfying all these properties. What is the probability that the number Clare wrote was Matej's favourite number?

Problem of the Week
Problem E and Solution
Favourite Number 3

Problem

Matej told his friend Clare the following properties about his favourite number:

- It is a positive seven-digit integer.
- It contains each of the digits from 1 through 7 exactly once.
- The first digit is not 1.
- The second digit is not 2.

Clare then wrote down a number satisfying all these properties. What is the probability that the number Clare wrote was Matej's favourite number?

Solution

Let $abcdefg$ represent any seven-digit number where a, b, c, d, e, f, and g are different digits from 1 through 7.

Note that this solution uses *factorial notation*, as in $4 \times 3 \times 2 \times 1 = 4!$. In general, if n is a positive integer, then $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. We now proceed with two different approaches to solving the problem.

Solution 1

In this solution we determine the number of possible numbers that Clare could have written down directly. Thus, Clare's number is of the form $abcdefg$ where $a \neq 1$ and $b \neq 2$. We consider the following two cases:

- **Case 1:** $a = 2$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 2$.

- **Case 2:** $a \neq 2$
  In this case, there are 5 choices for the digit a (since we also have $a \neq 1$). Once digit a is chosen, there are 5 choices for the digit b (since we have $b \neq 2$ and $b \neq a$), 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 25 \times 5!$ possibilities when $a \neq 2$.

Thus, the total number of possible numbers Claire could have written down is $6! + 25 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.

Solution 2

In this solution we determine the number of possible numbers that Clare could have written down indirectly.

First, we determine the total number of numbers of the form $abcdefg$ where a, b, c, d, e, f, and g are different digits from 1 through 7. There are 7 choices for the digit a, 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$ possibilities.

Next, we determine the number of possible seven-digit numbers of the form $abcdefg$ where $a = 1$ and/or $b = 2$. We consider the following two cases:

- **Case 1:** $a = 1$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 1$.

- **Case 2:** $a \neq 1$ and $b = 2$
  In this case, there are 5 choices for the digit a, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 5!$ possibilities when $a \neq 1$ and $b = 2$.

Thus, the total number of possible numbers Claire could have written down is $7! - 6! - 5 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.

Problem of the Week
Problem E
Coin Bags

Mara has a bag containing 2 gold coins and 2 silver coins. Sharad has a bag containing 2 gold coins, 2 silver coins, and b bronze coins (where $b > 1$). Mara and Sharad each randomly draw two coins from their bag without replacement. Note that drawing two coins *without replacement* means drawing two coins, one after the other, without putting the first coin back in the bag.

If the probability that the two coins are the same colour is the same for Mara and Sharad, determine the value of b.

Problem of the Week
Problem E and Solution
Coin Bags

Problem

Mara has a bag containing 2 gold coins and 2 silver coins. Sharad has a bag containing 2 gold coins, 2 silver coins, and b bronze coins (where $b > 1$). Mara and Sharad each randomly draw two coins from their bag without replacement. Note that drawing two coins *without replacement* means drawing two coins, one after the other, without putting the first coin back in the bag.

If the probability that the two coins are the same colour is the same for Mara and Sharad, determine the value of b.

Solution

First we consider Mara's bag, which contains a total of $2 + 2 = 4$ coins. There are 4 possible coins that can be drawn first, leaving 3 possible coins that can be drawn second. This gives a total of $4 \times 3 = 12$ ways of drawing two coins.

For both coins to be gold, there are 2 possible coins (either gold coin) that can be drawn first, and 1 coin that must be drawn second (the remaining gold coin). This gives a total of $2 \times 1 = 2$ ways of drawing two gold coins. For both coins to be silver, there are 2 possible coins that can be drawn first, and 1 coin that must be drawn second. This gives a total of $2 \times 1 = 2$ ways of drawing two silver coins. Therefore, Mara has a total of $2 + 2 = 4$ ways of drawing two coins of the same colour.

Thus, the probability of Mara drawing two coins of the same colour is $\frac{4}{12} = \frac{1}{3}$.

Next we consider Sharad's bag, which contains a total of $2 + 2 + b = b + 4$ coins. There are $(b + 4)$ possible coins that can be drawn first, leaving $(b + 3)$ possible coins that can be drawn second. This gives a total of $(b + 4)(b + 3)$ ways of drawing two coins.

As with Mara's bag, there are 2 ways of drawing two gold coins and 2 ways of drawing two silver coins. For both coins to be bronze, there are b possible coins that can be drawn first, and $(b - 1)$ possible coins that can be drawn second. This gives a total of $b(b - 1)$ ways of drawing two bronze coins. Therefore, Sharad has a total of $2 + 2 + b(b - 1) = b^2 - b + 4$ ways of drawing two coins of the same colour.

Thus, the probability of Sharad drawing two coins of the same colour is $\frac{b^2-b+4}{(b+4)(b+3)}$.

Since these probabilities are equal,

$$\begin{aligned}
\frac{1}{3} &= \frac{b^2 - b + 4}{(b+4)(b+3)} \\
(b+4)(b+3) &= 3(b^2 - b + 4) \\
b^2 + 7b + 12 &= 3b^2 - 3b + 12 \\
0 &= 2b^2 - 10b \\
0 &= b^2 - 5b \\
0 &= b(b - 5)
\end{aligned}$$

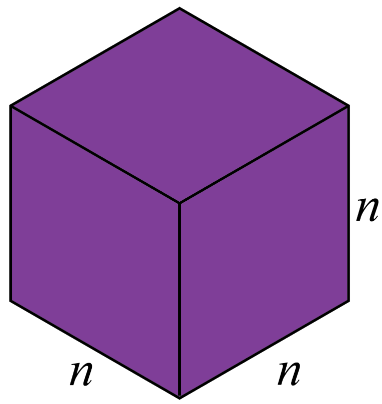
Since $b > 1$, it follows that $b = 5$.

Geometry & Measurement (G)

Take me to the cover

Problem of the Week
Problem E
Painting Faces

A cube has side length n, where n is a positive integer. Exactly three faces, which meet at a corner, are painted purple. The cube is then cut into n^3 smaller cubes with side length 1. If exactly 125 of these smaller cubes have no faces painted purple, then determine the value of n.



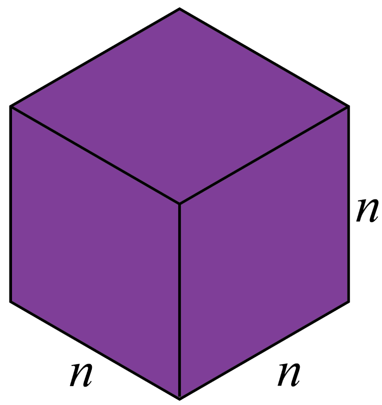
Problem of the Week

Problem E and Solution

Painting Faces

Problem

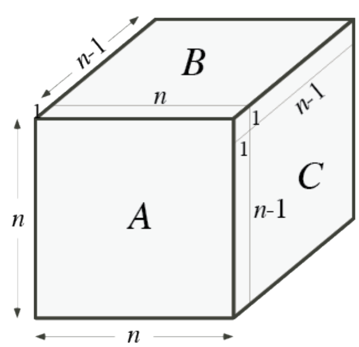
A cube has side length n, where n is a positive integer. Exactly three faces, which meet at a corner, are painted purple. The cube is then cut into n^3 smaller cubes with side length 1. If exactly 125 of these smaller cubes have no faces painted purple, then determine the value of n.



Solution

Solution 1

We label the faces painted purple as A, B, and C. We know that there are n^3 smaller cubes of side length 1. To determine the number of smaller cubes with all faces unpainted, we can subtract the number of cubes with some purple faces from the total number of cubes. Face A has dimensions n by n, and so contains n^2 unit cubes with some purple. Ignoring the area on face B which includes the smaller cubes also on face A, the remainder of face B has dimensions n by $(n-1)$, and so contains $n \times (n-1)$ more unit cubes with some purple. Ignoring the area on face C which includes the smaller cubes also on face A or on face B, face C has dimensions $(n-1)$ by $(n-1)$, and so contains $(n-1)(n-1)$ more unit cubes with some purple.



The number of unpainted cubes is thus, $n^3 - n^2 - n(n-1) - (n-1)(n-1)$. We can simplify this as

$$n^3 - n^2 - n(n-1) - (n-1)(n-1) = n^2(n-1) - n(n-1) - (n-1)(n-1)$$

Each term in this expression contains a common factor of $(n-1)$, so the expression simplifies to $(n-1)(n^2-n-(n-1)) = (n-1)(n^2-2n+1)$. This further simplifies to $(n-1)^3$. If the solver pauses here to think about this, if the unit cubes on face A then face B and finally face C are removed, we are left with a cube with side length $(n-1)$, and so $(n-1)^3$ unit cubes.

So, $(n-1)^3 = 125$, the actual number of unpainted cubes.

Taking the cube root, $n-1=5$, and so $n=6$ follows.

Solution 2

This solution uses the factor theorem which is generally taught in Grade 12.

As in Solution 1, the number of unpainted cubes is $n^3-n^2-n(n-1)-(n-1)(n-1)$. Setting this equal to 125, we have

$$\begin{aligned} n^3-n^2-n(n-1)-(n-1)(n-1) &= 125 \\ n^3-n^2-n^2+n-n^2+2n-1 &= 125 \\ n^3-3n^2+3n-126 &= 0 \end{aligned}$$

Let $f(n) = n^3-3n^2+3n-126$.

When $n=6$, $f(6) = 6^3-3(6^2)+3(6)-126 = 216-108+18-126 = 224-224 = 0$.

Since $f(6)=0$, $(n-6)$ is a factor of $f(n)$.

After long division (or synthetic division), we find $f(n) = (n-6)(n^2+3n+21)$.

Since $n^2+3n+21=0$ has no real solution, we know that $n=6$ is the only real solution to $n^3-3n^2+3n-126=0$.

Therefore, $n=6$.

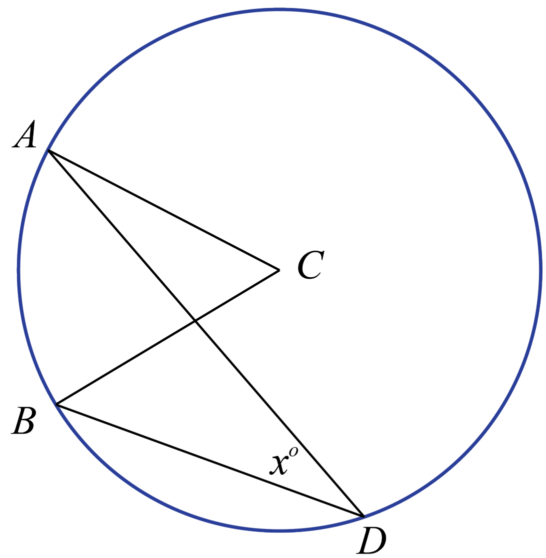
Problem of the Week

Problem E

Slice of an Arc

Points A, B, and D lie on the circumference of a circle with centre C.

If $\angle ADB = x^\circ$, then determine the measure of $\angle ACB$ in terms of x.

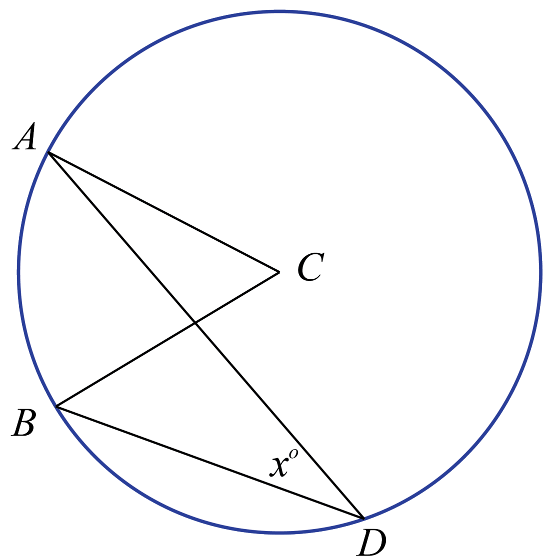


Problem of the Week
Problem E and Solution
Slice of an Arc

Problem

Points A, B, and D lie on the circumference of a circle with centre C.

If $\angle ADB = x^\circ$, then determine the measure of $\angle ACB$ in terms of x.

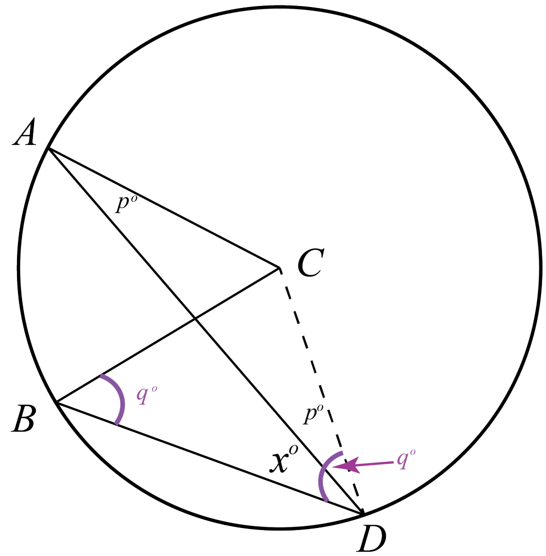


Solution

We construct radius CD. Let $\angle CAD = p^\circ$ and $\angle CBD = q^\circ$.

Since CA and CD are both radii of the circle, $CA = CD$. So $\triangle CAD$ is isosceles and $\angle CDA = \angle CAD = p^\circ$. Since the angles in a triangle add to 180°, $\angle ACD = (180 - 2p)^\circ$.

Since CB and CD are both radii of the circle, $CB = CD$. So $\triangle CBD$ is isosceles and $\angle CDB = \angle CBD = q^\circ$. Since the angles in a triangle add to 180°, $\angle BCD = (180 - 2q)^\circ$.



Now,

$$\begin{aligned}\angle ACB &= \angle ACD - \angle BCD\\ &= (180-2p)^\circ - (180-2q)^\circ\\ &= (2q-2p)^\circ\\ &= 2(q-p)^\circ\end{aligned}$$

Since $\angle CDB = \angle CDA + \angle ADB$, we have $q = p + x$.

Thus, $\angle ACB = 2(q-p)^\circ = 2x^\circ$.

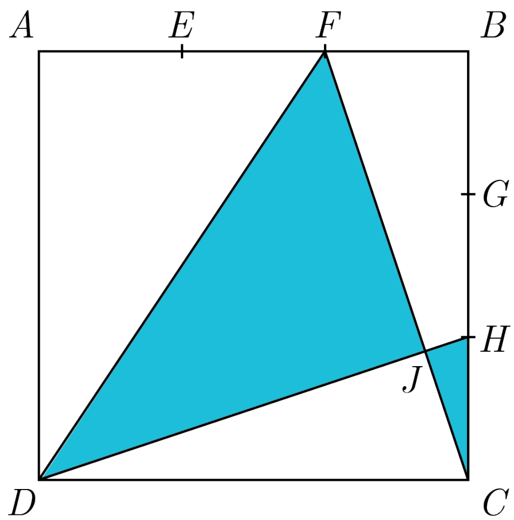
NOTE: In general, the angle inscribed at the centre of a circle is twice the size of the angle inscribed at the circumference by the same chord. That is, the angle inscribed by chord AB at the centre of the circle ($\angle ACB$) is double the angle inscribed by chord AB on the circumference ($\angle ADB$).

Problem of the Week

Problem E

A Fraction to Find

In square $ABCD$, points E and F lie on AB such that $AE = EF = FB = 10$. Similarly, points G and H lie on BC such that $BG = GH = HC = 10$. Let J be the intersection of line segments DH and CF. The areas of $\triangle DFJ$ and $\triangle CJH$ are then shaded.

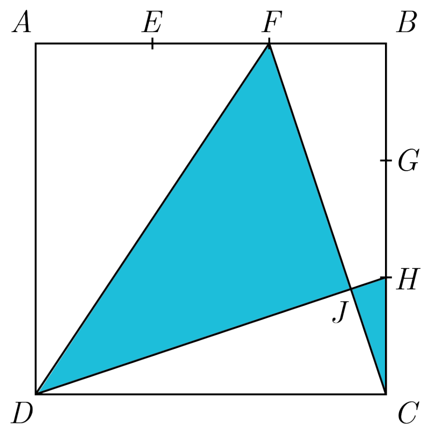


Determine the fraction of the area of square $ABCD$ that is shaded.

Problem of the Week
Problem E and Solution
A Fraction to Find

Problem

In square $ABCD$, points E and F lie on AB such that $AE = EF = FB = 10$. Similarly, points G and H lie on BC such that $BG = GH = HC = 10$. Let J be the intersection of line segments DH and CF. The areas of $\triangle DFJ$ and $\triangle CJH$ are then shaded.

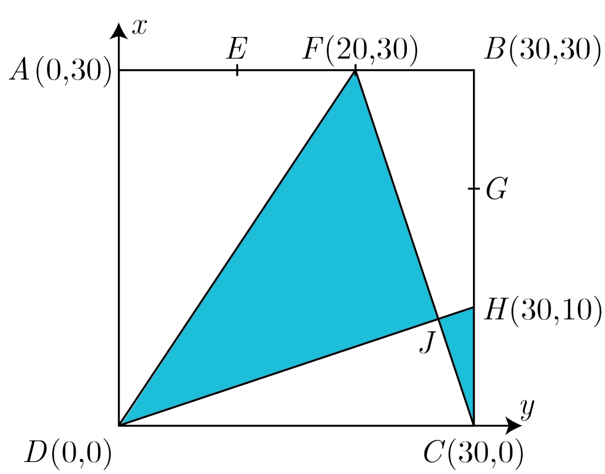


Determine the fraction of the area of square $ABCD$ that is shaded.

Solution

Solution 1

In this solution, we set square $ABCD$ on the Cartesian grid with vertex D at the origin. Then, since $AE = EF = FB = 10$, it follows that $AB = 30$, so the side length of $ABCD$ is 30. Thus, the coordinates of the vertices are $A(0, 30)$, $B(30, 30)$, $C(30, 0)$, and $D(0, 0)$. Since $AF = 20$ and AF is parallel to the x-axis, it follows that the coordinates of F are $(20, 30)$. Since $HC = 10$ and HC is parallel to the y-axis, it follows that the coordinates of H are $(30, 10)$.



Now we will find the coordinates of J by finding the point of intersection of the lines through DH and CF.

The line through DH has slope $\frac{10}{30} = \frac{1}{3}$ and y-intercept 0. Thus, its equation is $y = \frac{1}{3}x$.

The line through CF has slope $\frac{0-30}{30-20} = \frac{-30}{10} = -3$. To determine its y-intercept we substitute $(30, 0)$ into $y = -3x + b$. This gives $0 = -3(30) + b$, so $b = 90$. Thus, its equation is $y = -3x + 90$.

We set these two equations equal to each other to solve for the point of intersection.

$$\begin{aligned}\frac{1}{3}x &= -3x + 90\\ \frac{10}{3}x &= 90\\ x &= 90 \times \frac{3}{10} = 27\end{aligned}$$

Thus, $y = \frac{1}{3}(27) = 9$, so the coordinates of J are $(27, 9)$.

Since the slope of the line through DH is $\frac{1}{3}$ and the slope of the line through CF is -3, and these are negative reciprocals, it follows that these lines are perpendicular. Thus, both $\triangle DFJ$ and $\triangle CJH$ are right-angled triangles. So to determine the area of $\triangle DFJ$ we will consider the base to be DJ and the height to be JF. Using the Pythagorean theorem,

$$\begin{aligned}DJ &= \sqrt{27^2 + 9^2}\\ &= \sqrt{810}\\ &= 9\sqrt{10}\end{aligned}$$

$$\begin{aligned}JF &= \sqrt{(27-20)^2 + (9-30)^2}\\ &= \sqrt{7^2 + (-21)^2}\\ &= \sqrt{490} = 7\sqrt{10}\end{aligned}$$

Thus,

$$\text{Area } \triangle DFJ = \frac{1}{2} \times DJ \times JF = \frac{1}{2} \times 9\sqrt{10} \times 7\sqrt{10} = \frac{1}{2} \times 63 \times 10 = 315$$

To determine the area of $\triangle CJH$ we will consider the base to be $CH = 10$. To find the height, we drop a perpendicular from J to CH and call this point K. Then the height is $JK = 30 - 27 = 3$. Thus,

$$\text{Area } \triangle CJH = \frac{1}{2} \times CH \times JK = \frac{1}{2} \times 10 \times 3 = 15$$

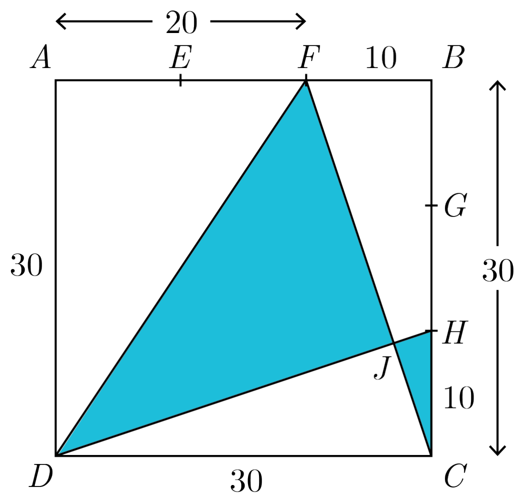
Then the total area shaded is $315 + 15 = 330$.

The area of square $ABCD$ is $30 \times 30 = 900$. Therefore, the fraction of this area that is shaded is $\frac{330}{900} = \frac{11}{30}$.

Solution 2

In this solution we use similar triangles to calculate the area indirectly.

Since $AE = EF = FB = 10$, and $ABCD$ is a square, it follows that $AB = BC = CD = AD = 30$. Since $AE = EF = 10$ it follows that $AF = 20$.



We can then determine the areas of right-angled triangles $\triangle DAF$ and $\triangle CBF$. The area of $\triangle DAF$ is $\frac{1}{2} \times 30 \times 20 = 300$ and the area of $\triangle CBF$ is $\frac{1}{2} \times 30 \times 10 = 150$.

Next we determine the area of $\triangle CJH$. First we notice that $\triangle CBF$ and $\triangle DCH$ are congruent, and $\triangle DCH$ has been rotated by 90°. Thus, $DH \perp CF$. Next we notice that $\angle HCJ = \angle BCF$ and $\angle CJH = \angle CBF = 90^\circ$. Thus, $\triangle CJH \sim \triangle CBF$.

Using the Pythagorean theorem, $CF = \sqrt{30^2 + 10^2} = \sqrt{1000} = 10\sqrt{10}$. Since the corresponding side in $\triangle CJH$ is $CH = 10$, it follows that the side lengths in $\triangle CBF$ are $\sqrt{10}$ times the length of their corresponding sides in $\triangle CJH$. Thus, the area of $\triangle CBF$ is $\sqrt{10} \times \sqrt{10} = 10$ times the area of $\triangle CJH$. Since the area of $\triangle CBF$ is 150, it follows that the area of $\triangle CJH$ is 15.

Finally, we can determine the total area shaded.

$$\begin{aligned}\text{Shaded area} &= \text{Area } ABCD - \text{Area } \triangle DAF - 2 \times \text{Area } \triangle CBF + 2 \times \text{Area } \triangle CJH \\ &= 30 \times 30 - 300 - 2 \times 150 + 2 \times 15 \\ &= 330\end{aligned}$$

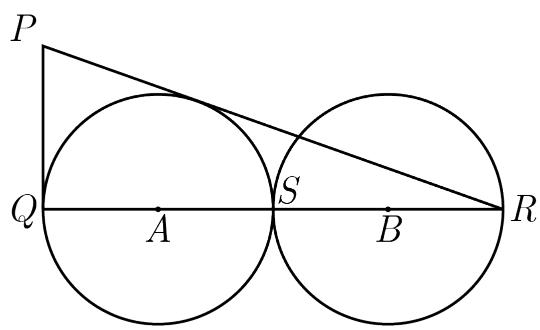
Therefore, the fraction of the area of square $ABCD$ that is shaded is $\frac{330}{900} = \frac{11}{30}$.

Problem of the Week

Problem E

A Triangle and Two Circles

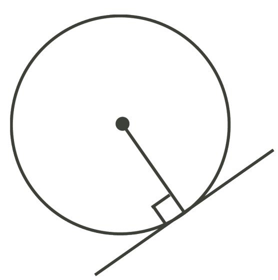
Two circles with centres A and B, each with a radius of 3, are tangent to each other at S. A straight line is drawn through A, S and B, meeting the circle with centre A at Q, $Q \neq S$, and the circle with centre B at R, $R \neq S$. Point P is then drawn so that PQ and PR are each tangent to the circle with centre A.



Determine the length of PQ.

Note: For this problem, you may want to use the following known results about circles:

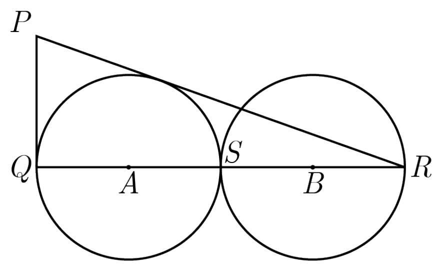
1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
2. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.



Problem of the Week
Problem E and Solution
A Triangle and Two Circles

Problem

Two circles with centres A and B, each with a radius of 3, are tangent to each other at S. A straight line is drawn through A, S and B, meeting the circle with centre A at Q, $Q \neq S$, and the circle with centre B at R, $R \neq S$. Point P is then drawn so that PQ and PR are each tangent to the circle with centre A.



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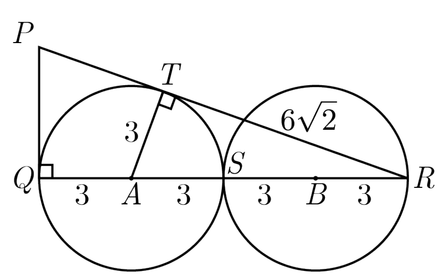
Solution

Since each circle has a radius of 3, $AQ = AS = BS = BR = 3$. From Result 1 given after the problem, since PQ is tangent to the circle with centre A, it follows that $PQ \perp QR$.

Let T be the point of tangency of the line PR. Then $AT \perp PR$. Since AT is a radius of the circle, $AT = 3$. Since $\triangle ATR$ is a right-angled triangle,
$TR^2 = AR^2 - AT^2 = 9^2 - 3^2 = 72$.
Thus, $TR = \sqrt{72} = 6\sqrt{2}$, since $TR > 0$.



We will now proceed with three different approaches.

Solution 1

In this solution we use trigonometry. Since $PQ \perp QR$ and $AT \perp PR$, $\triangle PQR$ and $\triangle ATR$ are right-angled triangles. In $\triangle PQR$, $\tan R = \frac{PQ}{QR} = \frac{PQ}{12}$. Similarly, in $\triangle ATR$, $\tan R = \frac{AT}{TR} = \frac{3}{6\sqrt{2}}$. Thus,

$$\frac{PQ}{12} = \frac{3}{6\sqrt{2}}$$

$$PQ = \frac{3}{6\sqrt{2}} \times 12 = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Therefore, the length of PQ is $3\sqrt{2}$.

Solution 2

In this solution we use similar triangles. Since $\angle PQR = \angle ATR = 90^\circ$ and $\angle PRQ = \angle ART$, it follows that $\triangle PQR \sim \triangle ATR$. Thus,

$$\frac{PQ}{QR} = \frac{AT}{TR}$$

$$\frac{PQ}{12} = \frac{3}{6\sqrt{2}}$$

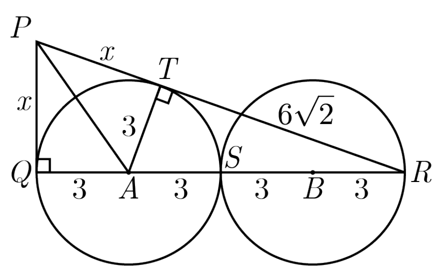
$$PQ = \frac{3}{6\sqrt{2}} \times 12 = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Therefore, the length of PQ is $3\sqrt{2}$.

Notice that while the initial approach is different, this solution is quite similar to Solution 1.

Solution 3

In this solution we use the Pythagorean Theorem. Draw line segment AP. Then $\triangle PQA$ and $\triangle PTA$ are right-angled triangles. Thus, $AP^2 = PQ^2 + AQ^2$ and $AP^2 = PT^2 + AT^2$. Since AQ and AT are both radii, $AQ = AT$. Thus, $PQ = PT$. Let $x = PQ = PT$.



Since $PQ \perp QR$, $\triangle PQR$ is right-angled with $PQ = x$, $QR = 12$, and $PR = x + 6\sqrt{2}$. Thus,

$$PR^2 = PQ^2 + QR^2$$

$$(x + 6\sqrt{2})^2 = x^2 + 12^2$$

$$x^2 + (12\sqrt{2})x + 72 = x^2 + 144$$

$$(12\sqrt{2})x = 72$$

$$x = \frac{72}{12\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

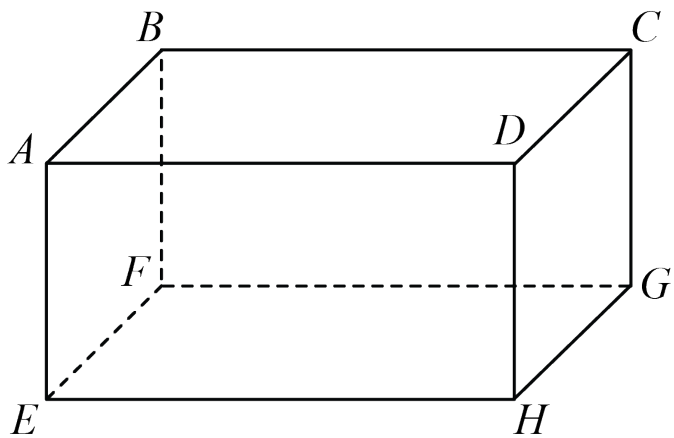
Therefore, the length of PQ is $3\sqrt{2}$.

Problem of the Week

Problem E

Corner Connection

In rectangular prism $ABCDEFGH$, the sum of the lengths of all of the edges is 28 cm, and the total surface area is 13 cm^2.



What is the length of EC, the diagonal of the prism?

Problem of the Week

Problem E and Solution

Corner Connection

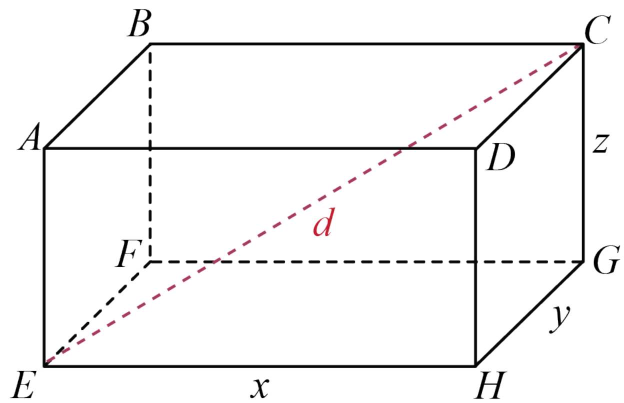
Problem

In rectangular prism $ABCDEFGH$, the sum of the lengths of all of the edges is 28 cm, and the total surface area is 13 cm^2.

What is the length of EC, the diagonal of the prism?

Solution

Let $EH = x$, $HG = y$, and $CG = z$. We construct EC and label it d.



By the Pythagorean Theorem in $\triangle EHG$, $EG^2 = EH^2 + HG^2$.
By the Pythagorean Theorem in $\triangle EGC$, $EC^2 = EG^2 + CG^2$.
Therefore, $EC^2 = EG^2 + CG^2 = EH^2 + HG^2 + CG^2$.
That is, $d^2 = x^2 + y^2 + z^2$.

Since the sum of the lengths of all the edges is 28, then $4x + 4y + 4z = 28$ or $x + y + z = 7$.
Since the surface area of the prism is 13, we know $2xy + 2yz + 2xz = 13$.

Since we have squared terms and pair factor terms it might be helpful to expand $(x + y + z)^2$.

$$\begin{aligned}(x+y+z)^2 &= (x+(y+z))^2 \\ &= x^2 + 2x(y+z) + (y+z)^2 \\ &= x^2 + 2xy + 2xz + y^2 + 2xy + z^2 \\ &= (x^2+y^2+z^2) + (2xy + 2xz + 2yz)\end{aligned}$$

Since $x + y + z = 7$, $d^2 = x^2 + y^2 + z^2$, and $2xy + 2yz + 2xz = 13$, we have

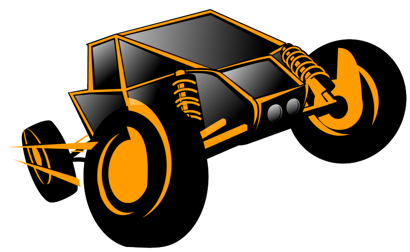
$$\begin{aligned}7^2 &= d^2 + 13 \\ d^2 &= 49 - 13 = 36\end{aligned}$$

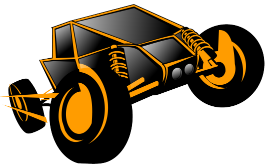
Since $d > 0$, we have $d = 6$. Therefore, the length of EC is 6 cm.

Problem of the Week
Problem E
Straight Ahead

Albertino and Mara are driving their remote control cars in a large field. Albertino's car starts 10 km north of Mara's car. At the same time, Albertino starts driving his car south and Mara start's driving her car east. Both cars travel at a constant speed of 40 km/hr.

After how many minutes of driving will the **total** distance travelled by the cars be equal to the distance between the cars?





Problem of the Week
Problem E and Solution
Straight Ahead

Problem

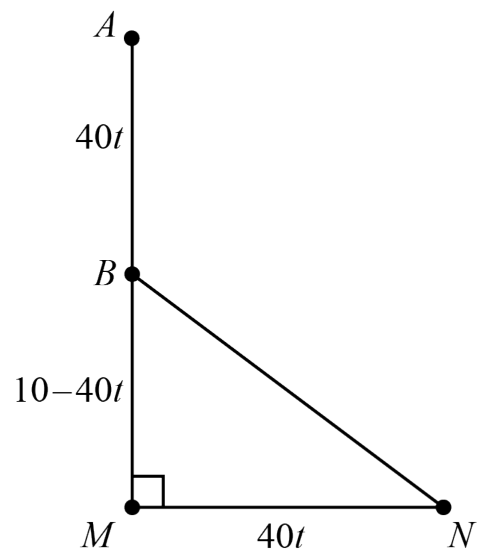
Albertino and Mara are driving their remote control cars in a large field. Albertino's car starts 10 km north of Mara's car. At the same time, Albertino starts driving his car south and Mara start's driving her car east. Both cars travel at a constant speed of 40 km/hr.

After how many minutes of driving will the **total** distance travelled by the cars be equal to the distance between the cars?

Solution

Let t be the time in hours until the total distance travelled by the cars is equal to the distance between the cars. Since each car is travelling at 40 km/hr, the distance traveled by each car in t hours is $40t$ km. We will now consider three cases: first when $40t < 10$, then when $40t = 10$, and finally when $40t > 10$.

- **Case 1:** $40t < 10$
  In this case, at time t, Albertino's car had not yet reached the initial position of Mara's car. The diagram shows the initial positions of Albertino's car, A, and Mara's car, M, as well as their positions after t hours of driving, which are represented as B and N, respectively. Point B is $40t$ km south of point A, and point N is $40t$ km east of point M. Since the initial distance between the cars, AM, is 10 km, it follows that $BM = (10 - 40t)$ km.



  Since each car travelled $40t$ km in t hours, the total distance travelled by the cars is $2 \times 40t = 80t$ km. Thus, we want to determine the value of t when $BN = 80t$. Since $\triangle BMN$ is a right-angled triangle, we can use the Pythagorean Theorem.

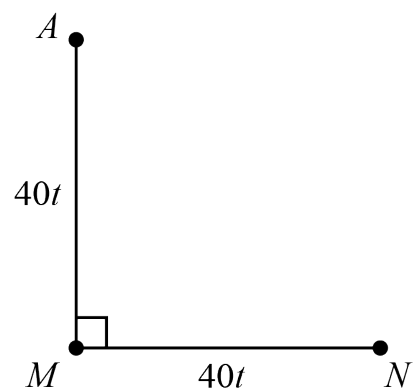
$$
\begin{aligned}
BN^2 &= BM^2 + MN^2 \\
(80t)^2 &= (10 - 40t)^2 + (40t)^2 \\
6400t^2 &= 100 - 800t + 1600t^2 + 1600t^2 \\
3200t^2 + 800t - 100 &= 0 \\
32t^2 + 8t - 1 &= 0
\end{aligned}
$$

Using the quadratic formula,

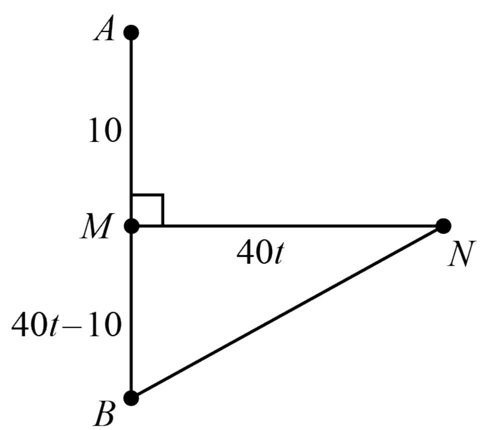
$$t = \frac{-8 \pm \sqrt{8^2 - 4(32)(-1)}}{2(32)} = \frac{-8 \pm \sqrt{192}}{64} = \frac{-1 \pm \sqrt{3}}{8}$$

Since $t > 0$, $t = \frac{-1+\sqrt{3}}{8} \approx 0.09150635$ hours. This is equivalent to approximately 5.49 minutes, or 5 minutes and 29 seconds.

- **Case 2:** $40t = 10$
  In this case, at time t, Albertino's car was at the initial position of Mara's car. So the distance between the cars is the distance that Mara's car travelled, which is $40t$ km. As before, the total distance travelled by the cars is $80t$ km. If these values are equal, then $40t = 80t$. The only solution to this is $t = 0$, but since $t > 0$, this solution is inadmissable. Thus, it is not possible that $40t = 10$.



- **Case 3:** $40t > 10$
  In this case, at time t, Albertino's car had travelled past the initial position of Mara's car. Since the distance between the initial positions of the cars is 10 km, and Albertino's car travelled $40t$ km, it follows that his car travelled $40t - 10$ km past the initial position of Mara's car.



  As before, the total distance travelled by the cars is $80t$ km. If this is equal to the distance between the cars, then $BN = 80t$. Then, in $\triangle BMN$, $BM + MN = 40t - 10 + 40t = 80t - 10 < 80t = BN$. Therefore, this does not satisfy the triangle inequality, so there is no solution for t. Thus, it is not possible that $40t > 10$.

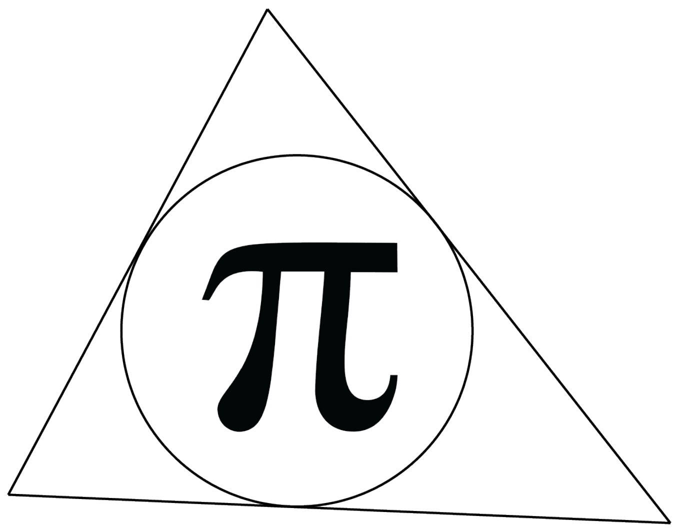
Therefore there is only one solution. After approximately 5.49 minutes, or 5 minutes and 29 seconds, the total distance travelled by the cars will be equal to the distance between the cars.

Problem of the Week

Problem E

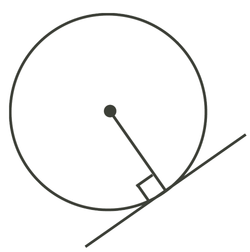
Pizza Pi Box

A box in the shape of a triangle with side lengths 13, 14, and 15 is being used to pack a pizza pi. The pizza is in the shape of a circle and just touches each of the three sides of the triangular box. What is the area of the circle?

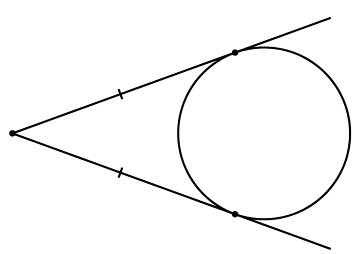


NOTE: For this problem, the following known results about circles may be useful:

- If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



- If two tangent segments are drawn to a circle from a point outside of the circle, then the lengths of those segments are equal.



Problem of the Week
Problem E and Solution
Pizza Pi Box

Problem

A box in the shape of a triangle with side lengths 13, 14, and 15 is being used to pack a pizza pi. The pizza is in the shape of a circle and just touches each of the three sides of the triangular box. What is the area of the circle?

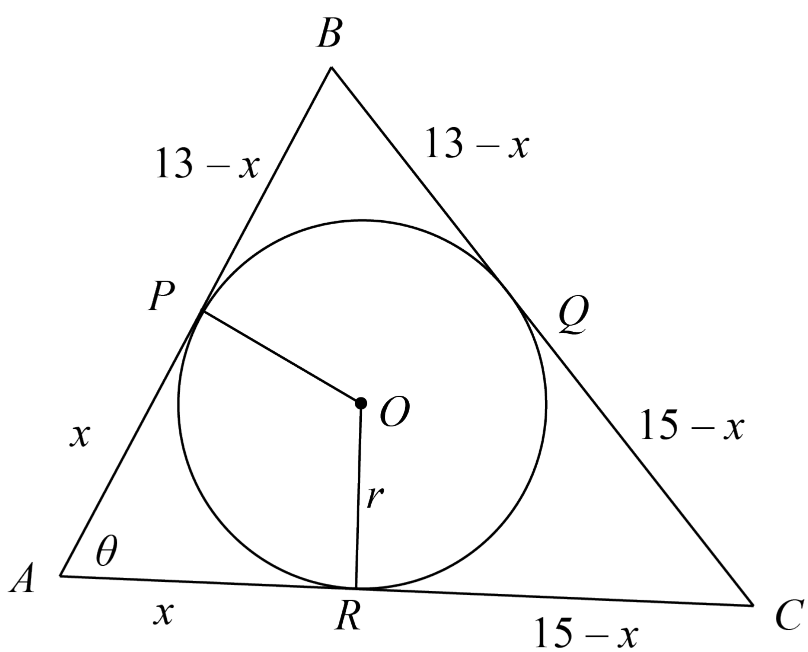
Solution

We label the triangle ABC, with $AB = 13$, $BC = 14$, and $AC = 15$.

Let P be the point where the circle touches the triangle on AB, Q be the point where the circle touches the triangle on BC, and R be the point where the circle touches the triangle on AC.

Let $\angle BAC = \theta$, and let the circle be centred at point O with radius r.

Using the second given result about circles, we know that $AP = AR$, $CR = CQ$, and $BP = BQ$. Let $AP = AR = x$. Then $BP = AB - AP = 13 - x$. Since $BP = BQ$, we have $BQ = BP = 13 - x$. Also, $CR = AC - AR = 15 - x$. Since $CR = CQ$, we have $CQ = 15 - x$.



Since $BC = 14$ and $BC = BQ + CQ$, we have $13 - x + 15 - x = 14$, thus $28 - 2x = 14$, and so $x = 7$.

Using the cosine law in $\triangle ABC$,

$$\begin{aligned}BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC)\\ 14^2 &= 13^2 + 15^2 - 2(13)(15)\cos(\theta)\\ \cos\theta &= \frac{14^2 - 13^2 - 15^2}{-2(13)(15)}\\ &= \frac{-198}{-390}\\ &= \frac{33}{65}\end{aligned}$$

Consider $\triangle AOR$ and $\triangle AOP$. Since $AR = AP$, $OR = OP = r$, and $AO = AO$ (same length), $\triangle AOR$ is congruent to $\triangle AOP$. Thus, $\angle OAP = \angle OAR = \dfrac{\theta}{2}$. Further, using the first given result about circles, we know that $\triangle AOR$ and $\triangle AOP$ are right-angled.

In $\triangle AOR$, $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{r}{x} = \dfrac{r}{7}$. Thus, $r = 7\tan\left(\dfrac{\theta}{2}\right)$.

Using the identities $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos\theta}{2}$ and $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{2}$, we obtain $\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{1+\cos\theta}{2} = \dfrac{49}{65}$, and $\sin^2\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{2} = \dfrac{16}{65}$.

Since $0 < \dfrac{\theta}{2} < 90^\circ$, $\cos\left(\dfrac{\theta}{2}\right) > 0$ and $\sin\left(\dfrac{\theta}{2}\right) > 0$. Thus, $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{7}{\sqrt{65}}$ and $\sin\left(\dfrac{\theta}{2}\right) = \dfrac{4}{\sqrt{65}}$.

Therefore, we have

$$\begin{aligned}r &= 7\tan\left(\frac{\theta}{2}\right)\\ &= \frac{7\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\\ &= \frac{7\left(\frac{4}{\sqrt{65}}\right)}{\frac{7}{\sqrt{65}}}\\ &= 4\end{aligned}$$

Therefore, the area of the circle is $\pi(4)^2 = 16\pi$.

Problem of the Week

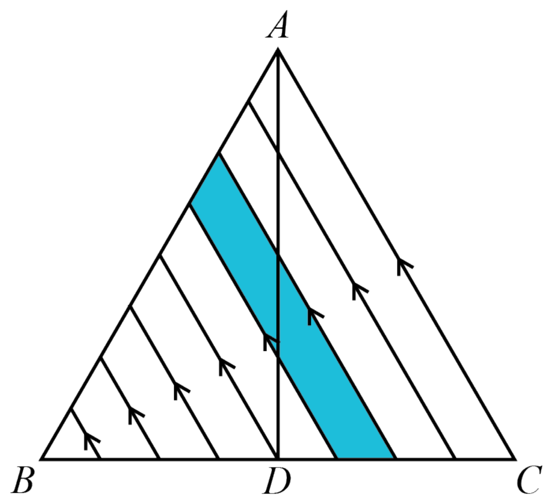
Problem E

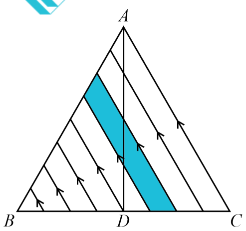
Yet Another Suncatcher

A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

- Sides AB and BC are each divided into 8 segments of equal length.
- Each point of division on AB is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

One of the sections is coloured blue, as shown. An altitude is constructed from A to D on BC, dividing the blue section into two parts. In the blue section, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.





Problem of the Week
Problem E and Solution
Yet Another Suncatcher

Problem

A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

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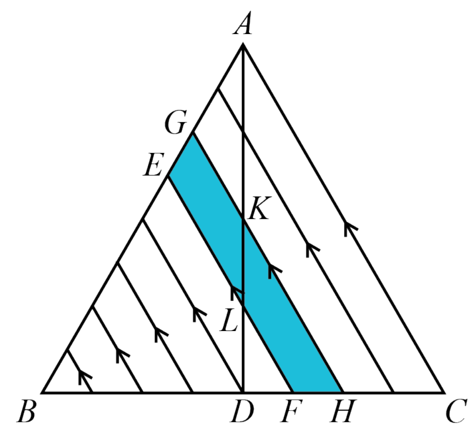
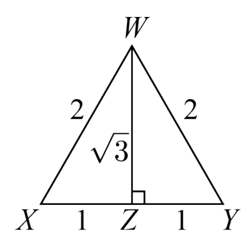
Solution

Solution 1



Since AB and BC are each divided into 8 equal segments, then each segment will have a length of $144 \div 8 = 18$ mm.

Label the blue section $EFHG$. Let K be the point of intersection between altitude AD and segment GH, and L be the point of intersection between altitude AD and segment EF. We want to find the ratio of the area of $GELK$ to the area of $KLFH$.



We first mention a few facts that we will use in our solution. For some equilateral $\triangle WXY$ with side length 2, altitude WZ bisects base XY and it follows that $XZ = ZY = 1$. Using the Pythagorean Theorem, the length of WZ is $\sqrt{2^2 - 1^2} = \sqrt{3}$. In any equilateral triangle, the ratio of the height to the side length is $\sqrt{3} : 2$. In other words, the height of any equilateral triangle is $\frac{\sqrt{3}}{2}$ times its side length. The altitude WZ also bisects $\angle XWY$, so $\angle XWZ = \angle YWZ = 30^\circ$ and $\triangle WXZ$ is a $30^\circ - 60^\circ - 90^\circ$ triangle whose sides are in the ratio $1 : \sqrt{3} : 2$.

Now going back to our problem, we will start by subtracting the area of $\triangle BEF$ from the area of $\triangle BGH$ to find the area of section $EFHG$.

We know that $\angle ABC = \angle BCA = \angle BAC = 60^\circ$. In $\triangle BEF$, $\angle EBF = \angle ABC = 60^\circ$ since they are the same angle. Since $EF \parallel AC$, $\angle BFE = \angle BCA = 60^\circ$ and $\angle BEF = \angle BAC = 60^\circ$. Since all of the angles in $\triangle BEF$ are 60°, it is an equilateral triangle with side length $5 \times 18 = 90$ mm. Using our above result, the height of $\triangle BEF = \frac{\sqrt{3}}{2} \times 90 = 45\sqrt{3}$. The area of $\triangle BEF = \frac{90 \times 45\sqrt{3}}{2} = 2025\sqrt{3}$ mm^2.

In a similar way, we can show that $\triangle BGH$ is equilateral with side length $6 \times 18 = 108$ mm. The height of $\triangle BGH = \frac{\sqrt{3}}{2} \times 108 = 54\sqrt{3}$ and the area of $\triangle BGH = \frac{108 \times 54\sqrt{3}}{2} = 2916\sqrt{3}$ mm^2. Thus,

$$\begin{aligned}\text{area of } EFHG &= \text{area of } \triangle BGH - \text{area of } \triangle BEF \\ &= 2916\sqrt{3} - 2025\sqrt{3} = 891\sqrt{3} \text{ mm}^2\end{aligned}$$

Next, we find the area of $KLFH$ by finding the area of $\triangle LDF$ and subtracting it from the area of $\triangle KDH$.

In $\triangle LDF$, $\angle LDF = \angle ADC = 90^\circ$, since the altitude AD is perpendicular to base BC and the two angles represent the same angle. Since $EF \parallel AC$, $\angle DFL = \angle BCA = 60^\circ$. It follows that $\angle DLF = 30^\circ$. Therefore, $\triangle LDF$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $DF : DL : LF = 1 : \sqrt{3} : 2$. Since $DF = 18$, it follows that $DL = 18\sqrt{3}$. The area of $\triangle LDF = \frac{18 \times 18\sqrt{3}}{2} = 162\sqrt{3}$ mm^2.

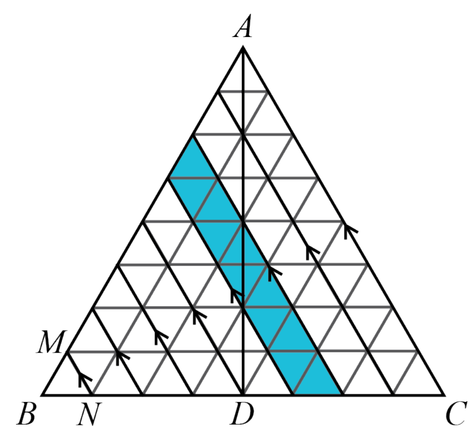
In a similar way, we can show that $\triangle KDH$ is a $30^\circ - 60^\circ - 90^\circ$ triangle with $DH : DK : KH = 1 : \sqrt{3} : 2$. Since $DH = 2 \times 18 = 36$, it follows that $DK = 36\sqrt{3}$. The area of $\triangle KDH = \frac{36 \times 36\sqrt{3}}{2} = 648\sqrt{3}$ mm^2. Then,

$$\text{area of } KLFH = \text{area of } \triangle KDH \; - \text{area of } \triangle LDF = 648\sqrt{3} - 162\sqrt{3} = 486\sqrt{3} \text{ mm}^2$$
$$\text{area of } GELK = \text{area of } EFHG \; - \text{area of } KLFH = 891\sqrt{3} - 486\sqrt{3} = 405\sqrt{3} \text{ mm}^2$$

Therefore, the ratio of the area of $GELK$ to the area of $KLFH$ is $405\sqrt{3} : 486\sqrt{3}$ or $5 : 6$.

Solution 2

Label the endpoints of the line segment closest to B as M and N. Observe that $\triangle ABC$ can be tiled with small equilateral triangles congruent to $\triangle BMN$. That is, equilateral triangles with side length 18 mm. A complete justification of this is not provided here but you may wish to verify this for yourself.



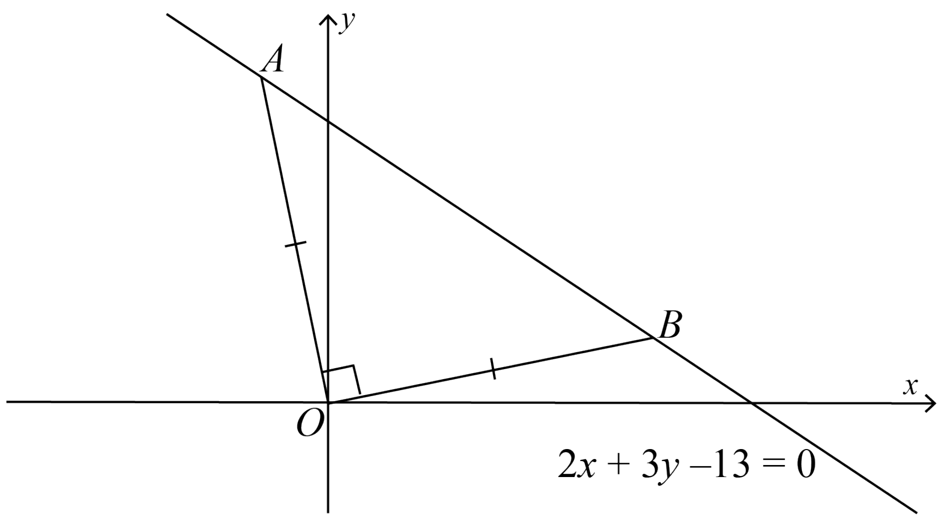
In the second section from B there are 3 small equilateral triangles. In the third section there are 5 small equilateral triangles. In the fourth section there are 7 small equilateral triangles, and so on. The blue section corresponds to the sixth section and contains 11 small equilateral triangles.

Of these 11 small equilateral triangles, there are $4 + \frac{1}{2} + \frac{1}{2} = 5$ on the left of altitude AD. A complete justification of this is not provided here, but you may wish to verify this yourself. Thus, there are $11 - 5 = 6$ on the right of altitude AD. Note that each of these small equilateral triangles have the same area. Therefore, the ratio of the blue area on the left side of the altitude to the blue area on the right side of the altitude is $5 : 6$.

Problem of the Week
Problem E
Tilted Triangle

Suppose that $\triangle OAB$ is such that vertex O is located at the origin, and vertices A and B lie on the line $2x + 3y - 13 = 0$ with $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.

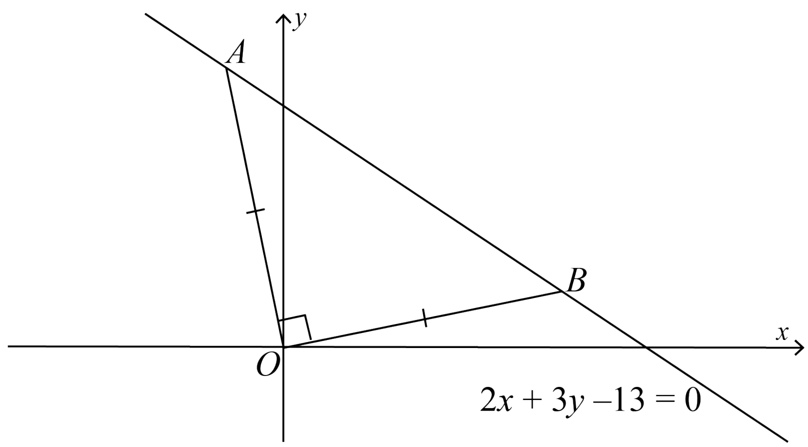


Problem of the Week
Problem E and Solution
Tilted Triangle

Problem

Suppose that $\triangle OAB$ is such that vertex O is located at the origin, and vertices A and B lie on the line $2x + 3y - 13 = 0$ with $\angle AOB = 90°$ and $OA = OB$.

Determine the area of $\triangle OAB$.



Solution

By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$.

Since points A and B lie on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$.

Since $OA = OB$, we have

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
13a^2 - 13b^2 - 52a + 52b &= 0 \\
a^2 - b^2 - 4a + 4b &= 0
\end{aligned}$$

By factoring, we then obtain $(a + b)(a - b) - 4(a - b) = 0$ or $(a - b)(a + b - 4) = 0$.

This implies that, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$.

From here we proceed with two different solutions.

Solution 1

Since B has coordinates $\left(b, \frac{-2b+13}{3}\right)$, the slope of OB is equal to $\dfrac{\frac{-2b+13}{3}}{b} = \dfrac{-2b+13}{3b}$.

Since A has coordinates $\left(a, \frac{-2a+13}{3}\right)$, the slope of OA is equal to $\dfrac{\frac{-2a+13}{3}}{a} = \dfrac{-2a+13}{3a}$.

Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of OA is the negative reciprocal of the slope of OB. Therefore,

$$\begin{aligned} \frac{-2a+13}{3a} &= -\frac{1}{\frac{-2b+13}{3b}} \\ &= \frac{-3b}{-2b+13} \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} (-2a+13)(-2b+13) &= (-3b)(3a) \\ 4ab - 26a - 26b + 169 &= -9ab \\ 13ab - 26a - 26b + 169 &= 0 \\ ab - 2a - 2b + 13 &= 0 \end{aligned}$$

Substituting $a = 4 - b$, we have

$$\begin{aligned} (4-b)b - 2(4-b) - 2b + 13 &= 0 \\ 4b - b^2 - 8 + 2b - 2b + 13 &= 0 \\ b^2 - 4b - 5 &= 0 \end{aligned}$$

Factoring, this becomes $(b-5)(b+1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$.

In each case, the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

Since $\triangle AOB$ is a right-angled triangle, we can use OB as the base and OA as the height in the formula for the area of a triangle.

Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26} \times \sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Solution 2

Since $a = 4 - b$, we can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is right-angled at O, by the Pythagorean Theorem, $AB^2 = OA^2 + OB^2$. Since $OA = OB$, this can be written $AB^2 = 2OB^2$. Thus,

$$\begin{aligned}
(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left(b^2 + \left(\frac{-2b+13}{3}\right)^2\right) \\
(2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left(b^2 + \frac{4b^2-52b+169}{9}\right) \\
4b^2 - 16b + 16 + \frac{16b^2-64b+64}{9} &= 2b^2 + \frac{8b^2-104b+338}{9}
\end{aligned}$$

Multiplying by 9 and simplifying, we obtain

$$\begin{aligned}
36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
26b^2 - 104b - 130 &= 0 \\
b^2 - 4b - 5 &= 0
\end{aligned}$$

Factoring, this becomes $(b-5)(b+1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

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In each case, the length of OB is $\sqrt{5^2+1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

Since $\triangle AOB$ is a right-angled triangle, we can use OB as the base and OA as the height in the formula for the area of a triangle.

Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26} \times \sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Number Sense (N)

Take me to the cover

Problem of the Week

Problem E

Sticky Notes

The students in an art class displayed their final projects for an art exhibit at their school. Some of the projects were set up in the gym and the rest were in the library. After the exhibit finished, each student in the class went around and put a sticky note with a positive comment on each of their classmate's projects. Once they were done, the projects in the gym had a total of 308 sticky notes and the projects in the library had a total of 198 sticky notes. How many students were in the art class? How many of their projects were in the library?

Problem of the Week

Problem E and Solution

Sticky Notes

Problem

The students in an art class displayed their final projects for an art exhibit at their school. Some of the projects were set up in the gym and the rest were in the library. After the exhibit finished, each student in the class went around and put a sticky note with a positive comment on each of their classmate's projects. Once they were done, the projects in the gym had a total of 308 sticky notes and the projects in the library had a total of 198 sticky notes. How many students were in the art class? How many of their projects were in the library?

Solution

Let n represent the number of students in the art class. Each student wrote $n-1$ sticky notes, so in total there were $n(n-1)$ sticky notes. We know the total number of sticky notes is $308 + 198 = 506$. Thus,

$$\begin{aligned} n(n-1) &= 506 \\ n^2 - n &= 506 \\ n^2 - n - 506 &= 0 \\ (n-23)(n+22) &= 0 \end{aligned}$$

Therefore $n = 23$ or $n = -22$. Since n represents the number of students, it follows that $n > 0$, so we can conclude that $n = 23$. Then there were 23 students in the class and they each wrote 22 sticky notes for their classmates' projects and each received 22 sticky notes on their own project.

Since there were a total of 198 sticky notes in the library, and each project had 22 sticky notes, it follows that there were $198 \div 22 = 9$ projects in the library.

Therefore, there were 23 students in the art class and 9 of their projects were in the library.

Problem of the Week

Problem E

To the Peak with Powers

Avital can choose four different numbers w, x, y, and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

$$w^x + y^z$$

$w^x + y^z$

Problem of the Week
Problem E and Solution
To the Peak with Powers

Problem

Avital can choose four different numbers w, x, y, and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

Solution

Consider w^x, where w and x are different numbers from $\{-1, -2, -3, -4, -5\}$. What is the largest possible value for w^x?

Since x will be negative, we write $w^x = \frac{1}{w^{-x}}$, where $-x > 0$.

If x is odd, then since w is negative, then w^x will be negative.
If x is even, then w^x will be positive.

So to make w^x as large as possible, we make x even. So $x = -2$ or $x = -4$.

Also, in order to make $w^x = \frac{1}{w^{-x}}$ as large as possible, we want to make the denominator, w^{-x}, as small as possible, so w should be as small as possible in absolute value.

Therefore, the largest possible value of w^x will be when $w = -1$ and $x = -2$ or $x = -4$. In both cases, $w^x = 1$, since $(-1)^{-2} = (-1)^{-4} = 1$.

What is the second largest possible value for w^x?

Again, we need x to be even to make w^x positive, and from above, we can assume that $w \neq -1$.

When $x = -2$, the smallest possible base (in absolute value) is $w = -3$ and $w^x = \frac{1}{(-3)^2} = \frac{1}{9}$.

When $x = -4$, the smallest possible base (in absolute value) is $w = -2$ and $w^x = \frac{1}{(-2)^4} = \frac{1}{16}$.

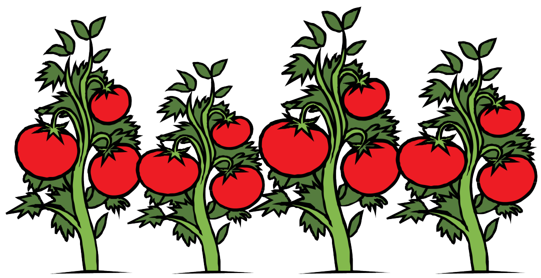
The largest of these two values for w^x is $\frac{1}{9}$.

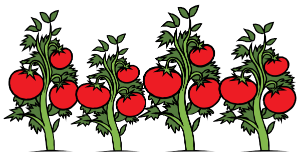
Therefore, the two largest possible values for w^x are 1 and $\frac{1}{9}$.

Thus, the largest possible value of $w^x + y^z$ is $1 + \frac{1}{9} = \frac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$. This will occur when $w = -1$, $x = -4$, $y = -3$, and $z = -2$ or when $w = -3$, $x = -2$, $y = -1$, and $z = -4$.

Problem of the Week
Problem E
Multiplying Tomatoes 2

Nia has four tomato plants, each with a different number of tomatoes on it. The product of the number of tomatoes on each plant is 24 300. The total number of tomatoes on the three plants with the most tomatoes is 43, and the total number of tomatoes on the three plants with the fewest tomatoes is 34. Determine all possibilities for the number of tomatoes on each plant.





Problem of the Week

Problem E and Solution

Multiplying Tomatoes 2

Problem

Nia has four tomato plants, each with a different number of tomatoes on it. The product of the number of tomatoes on each plant is 24 300. The total number of tomatoes on the three plants with the most tomatoes is 43, and the total number of tomatoes on the three plants with the fewest tomatoes is 34. Determine all possibilities for the number of tomatoes on each plant.

Solution

This problem is asking us to find four distinct positive integers with a product of 24 300, such that the sum of the smallest three integers is 34 and the sum of the largest three integers is 43. We start by letting the four integers be a, b, c, and d. Since the sum of the smallest three integers is 34, it follows that $a + b + c = 34$. Since the sum of the largest three integers is 43, it follows that $b + c + d = 43$. Subtracting these two equations gives $d - a = 9$.

It would help if we had an upper bound for the smallest integer. Suppose $a = 12$. Then $d = 12 + 9 = 21$. The smallest product of four such distinct integers is $12 \times 13 \times 14 \times 21 = 45\,864$. Since this is larger than 24 300, it follows that $a < 12$.

Similarly, it would help to have a lower bound for the smallest integer. Suppose $a = 6$. Then $d = 6 + 9 = 15$. The largest product of four such distinct integers is $6 \times 13 \times 14 \times 15 = 16\,380$. Since this is smaller than 24 300, it follows that $a > 6$.

This leaves us with the following possibilities for (a, d): $(7, 16)$, $(8, 17)$, $(9, 18)$, $(10, 19)$, or $(11, 20)$. Next we can factor 24 300 to obtain $24\,300 = 2^2 \times 3^5 \times 5^2$. From the factorization we can eliminate some of these pairs. Since 7, 17, 19, and 11 are not factors of 24 300, they cannot be any of the four integers. Thus, the only possibility left is $(a, d) = (9, 18)$.

Since $24\,300 = 2^2 \times 3^5 \times 5^2$, it follows that the middle two integers must have a product of $2 \times 3 \times 5^2$. The only possibilities between 9 and 18 are 10 and 15.

Therefore, there is only one possibility. The number of tomatoes on each plant is 9, 10, 15, and 18.

Problem of the Week

Problem E

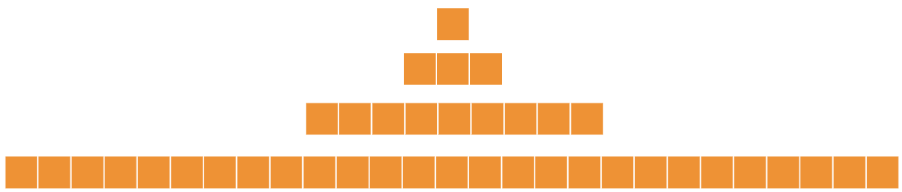
Power Trail

The first four terms of a geometric sequence are

$$a,\ b,\ 4a,\ a-6+b$$

for some numbers a and b with $a \neq 0$.

Determine all possibilities for the values of a and b.



NOTE: A geometric sequence is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 2, 6, 18 is a geometric sequence with three terms and a common ratio of 3. You can explore geometric sequences further in our CEMC Courseware.

Problem of the Week
Problem E and Solution
Power Trail

Problem

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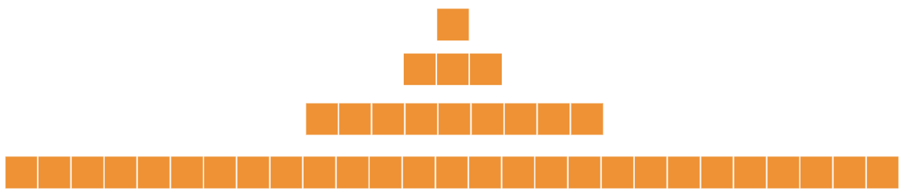

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Solution

In a geometric sequence with first term t_1, second term t_2, third term t_3, and fourth term t_4, we have $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3} = \dfrac{t_3}{t_4}$.

Here, $t_1 = a$, $t_2 = b$, $t_3 = 4a$, and $t_4 = a - 6 + b$.

Therefore, from $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3}$, we have

$$\begin{aligned} \frac{a}{b} &= \frac{b}{4a} \\ 4a^2 &= b^2 \\ 2a &= \pm b \end{aligned}$$

There are now two cases to consider.

- **Case 1:** When $b = 2a$, then the sequence can be written as a, $2a$, $4a$, $3a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned} \frac{a}{2a} &= \frac{4a}{3a-6} \\ 3a^2 - 6a &= 8a^2 \\ 0 &= 5a^2 + 6a \\ &= a(5a+6) \end{aligned}$$

  Thus, $a = 0$ or $a = -\frac{6}{5}$. Since $a \neq 0$, we have $a = -\frac{6}{5}$.

  Then $b = 2a = 2(-\frac{6}{5}) = -\frac{12}{5}$ and $a - 6 + b = -\frac{6}{5} - 6 + (-\frac{12}{5}) = -\frac{48}{5}$

  Therefore, the sequence, a, b, $4a$, $a - 6 + b$ is $-\frac{6}{5}$, $-\frac{12}{5}$, $-\frac{24}{5}$, $-\frac{48}{5}$.

  This sequence is indeed geometric, with first term $-\frac{6}{5}$ and common ratio 2.

- **Case 2:** When $b = -2a$, then the sequence can be written as a, $-2a$, $4a$, $-a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned} \frac{a}{-2a} &= \frac{4a}{-a-6} \\ -a^2 - 6a &= -8a^2 \\ 7a^2 - 6a &= 0 \\ a(7a-6) &= 0 \end{aligned}$$

  Thus, $a = 0$ or $a = \frac{6}{7}$. Since $a \neq 0$, we have $a = \frac{6}{7}$.

  Then $b = -2a = -2(\frac{6}{7}) = -\frac{12}{7}$ and $a - 6 + b = \frac{6}{7} - 6 + (-\frac{12}{7}) = -\frac{48}{7}$

  Therefore, the sequence, $a, b, 4a, a - 6 + b$ is $\frac{6}{7}$, $-\frac{12}{7}$, $\frac{24}{7}$, $-\frac{48}{7}$.

  This sequence is indeed geometric, with first term $\frac{6}{7}$ and common ratio -2.

Therefore, there are two possibilities. It could be the case that $a = -\frac{6}{5}$ and $b = -\frac{12}{5}$, or it could be the case that $a = \frac{6}{7}$ and $b = -\frac{12}{7}$.

Problem of the Week

Problem E

Math Test Math 3

Theo has had three tests in his math class so far. He does the following calculations using the average (mean) of some of his marks.

- First he calculates the average of his first and second test marks. Then he calculates the average of this value with his third test mark to get 74%.
- First he calculates the average of his second and third test marks. Then he calculates the average of this value with his first test mark to get 73.5%.
- First he calculates the average of his first and third test marks. Then he calculates the average of this value with his second test mark to get 76%.

Determine the average his three test marks.

Problem of the Week
Problem E and Solution
Math Test Math 3

Problem

Theo has had three tests in his math class so far. He does the following calculations using the average (mean) of some of his marks.

- First he calculates the average of his first and second test marks. Then he calculates the average of this value with his third test mark to get 74%.
- First he calculates the average of his second and third test marks. Then he calculates the average of this value with his first test mark to get 73.5%.
- First he calculates the average of his first and third test marks. Then he calculates the average of this value with his second test mark to get 76%.

Determine the average his three test marks.

Solution

Let a, b, and c represent the test marks, in order. We are looking for the average (mean) of a, b, and c, which is $\frac{a+b+c}{3}$.

The average of his first and second test marks is $\frac{a+b}{2}$. The average of this value with his third test mark is $\frac{\frac{a+b}{2}+c}{2}$. We know this is equal to 74. Thus,

$$\begin{aligned}\frac{\frac{a+b}{2}+c}{2} &= 74\\ \frac{a+b}{2}+c &= 148\\ a+b+2c &= 296 \qquad (1)\end{aligned}$$

Similarly, we can write the following two equations:

$$\begin{aligned}\frac{\frac{b+c}{2}+a}{2} &= 73.5\\ \frac{b+c}{2}+a &= 147\\ b+c+2a &= 294 \qquad (2)\end{aligned}$$

$$\begin{aligned}\frac{\frac{a+c}{2}+b}{2} &= 76\\ \frac{a+c}{2}+b &= 152\\ a+c+2b &= 304 \qquad (3)\end{aligned}$$

Next we add equations (1), (2), and (3):

$$\begin{aligned}(a+b+2c)+(b+c+2a)+(a+c+2b) &= 296+294+304\\ 4a+4b+4c &= 894\\ a+b+c &= 223.5\\ \frac{a+b+c}{3} &\approx 74.5\end{aligned}$$

Therefore, the average of his three test marks is 74.5%.

Extension: You may have noticed that the average of computed values 74, 73.5, and 76 is also equal to 74.5, the average of the test marks. In general, this will be the case for any three test results, when computed this way. Can you convince yourself of this?

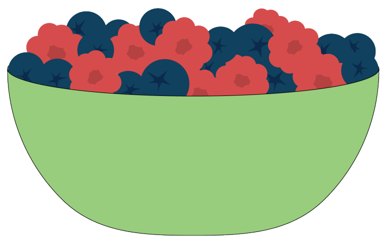
Problem of the Week

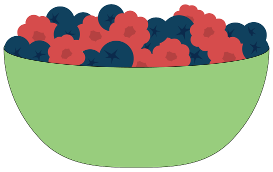
Problem E

Fruit Bowl

A bowl contains raspberries and blueberries. Natacha added 10 more raspberries to the bowl, then determined that $\frac{3}{10}$ of the berries in the bowl were raspberries. Jing then added 20 more blueberries to the bowl, then determined that $\frac{1}{4}$ of the berries in the bowl were raspberries.

What fraction of the original berries in the bowl were raspberries?





Problem of the Week

Problem E and Solution

Fruit Bowl

Problem

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What fraction of the original berries in the bowl were raspberries?

Solution

Let r represent the number of raspberries originally in the bowl, and let b represent the number of blueberries originally in the bowl.

After Natacha added 10 more raspberries, there were $r + 10$ raspberries and b blueberries. Since $\frac{3}{10}$ of the berries in the bowl were raspberries,

$$\begin{aligned} \frac{r+10}{r+10+b} &= \frac{3}{10} \\ 10r + 100 &= 3r + 30 + 3b \\ 7r - 3b + 70 &= 0 \qquad (1) \end{aligned}$$

After Jing added 20 more blueberries, there were $r + 10$ raspberries and $b + 20$ blueberries. Since $\frac{1}{4}$ of the berries in the bowl were raspberries,

$$\begin{aligned} \frac{r+10}{r+10+b+20} &= \frac{1}{4} \\ 4r + 40 &= r + b + 30 \\ 3r + 10 &= b \qquad (2) \end{aligned}$$

Substituting equation (2) into equation (1),

$$\begin{aligned} 7r - 3(3r + 10) + 70 &= 0 \\ 7r - 9r - 30 + 70 &= 0 \\ -2r + 40 &= 0 \\ r &= 20 \end{aligned}$$

Then $b = 3(20) + 10 = 70$. Therefore, there were originally 20 raspberries and 70 blueberries in the bowl. Thus, the fraction of the original berries in the bowl that were raspberries is $\frac{20}{20+70} = \frac{20}{90} = \frac{2}{9}$.

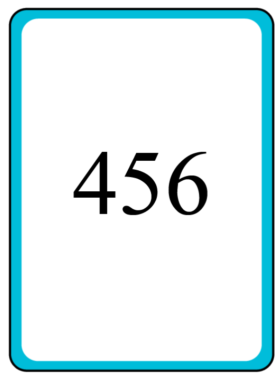
Problem of the Week

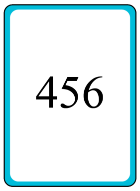
Problem E

A New Deck of Cards

Joanna has a deck of cards. Each card in the deck has a three-digit positive integer on it, and there is exactly one card in the deck for every three-digit positive integer.

Joanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.





Problem of the Week
Problem E and Solution
A New Deck of Cards

Problem

Joanna has a deck of cards. Each card in the deck has a three-digit positive integer on it, and there is exactly one card in the deck for every three-digit positive integer.

Joanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.

Solution

To begin, we determine the number of cards in the deck. There are 999 positive integers less than 1000. Of these, 90 are two-digit integers and 9 are single-digit integers. Therefore, there are $999 - 90 - 9 = 900$ three-digit positive integers, and so 900 cards in the deck.

Next, we determine the digit combinations that sum to 15.

- **Case 1:** One of the digits on the card is a 0.
  Then the other two digits on the card must add to 15. There are two possibilities for the digits. The three digits must be $0, 6, 9$ or $0, 7, 8$.

- **Case 2:** One of the digits on the card is a 1, but the number does not contain a 0.
  Then the other two digits on the card must add to 14. There are three possibilities for the digits. The digits must be $1, 5, 9$ or $1, 6, 8$ or $1, 7, 7$.

- **Case 3:** One of the digits on the card is a 2, but the number does not contain a 0 or 1.
  Then the other two digits on the card must add to 13. There are three possibilities for the digits. The digits must be $2, 4, 9$ or $2, 5, 8$ or $2, 6, 7$.

- **Case 4:** One of the digits on the card is a 3, but the number does not contain a 0, 1, or 2.
  Then the other two digits on the card must add to 12. There are four possibilities for the digits. The digits must be $3, 3, 9$ or $3, 4, 8$ or $3, 5, 7$ or $3, 6, 6$.

- **Case 5:** One of the digits on the card is a 4, but the number does not contain a 0, 1, 2, or 3.
  Then the other two digits on the card must add to 11. There are two possibilities for the digits. The digits must be $4, 4, 7$ or $4, 5, 6$.

- **Case 6:** One of the digits on the card is a 5, but the number does not contain a 0, 1, 2, 3, or 4.
  Then the other two digits on the card must add to 10. There is only one possibility for the digits. The digits must be $5, 5, 5$.

Now that we know what combinations of digits can be on the cards, we can determine the number of cards that can be created from each combination.

- **Case (a):** One of the digits on the card is 0.
  Earlier we found that there were two such digit combinations: $0, 6, 9$ and $0, 7, 8$. This is a special case since 0 cannot appear in the number as the hundreds digit for the number to be a three-digit number. For each of the two digit combinations, the 0 can be placed in two ways, in the tens digit or the units digit. Once the 0 is placed, the other two numbers can be placed in the remaining two spots in two ways. Thus, each digit combination can form $2 \times 2 = 4$ three-digit numbers. Since there are two digit combinations, there are $2 \times 4 = 8$ cards in the deck that contain a 0 whose digits add to 15.

- **Case (b):** All three digits on the card are different and the number does not contain a 0.
  From the earlier cases, there are eight such combinations: $1, 5, 9$, and $1, 6, 8$, and $2, 4, 9$, and $2, 5, 8$, and $2, 6, 7$, and $3, 4, 8$, and $3, 5, 7$, and $4, 5, 6$. For each of these combinations, the hundreds digit can be placed in three ways. For each of these three choices for hundreds digit, the tens digit can be placed in two ways. Once the hundreds digit and tens digit are selected, the units digit must get the third number. So each combination can form $3 \times 2 = 6$ different numbers. Since there are eight digit combinations, there are $8 \times 6 = 48$ cards in the deck that contain three different digits other than 0 whose digits add to 15.

- **Case (c):** Two of the digits on the card are the same and the number does not contain a 0.
  From the earlier cases, there are four such combinations: $1, 7, 7$, and $3, 3, 9$, and $3, 6, 6$, and $4, 4, 7$. For each of these combinations, the unique number can be placed in one of three spots. Once the unique number is placed the other two numbers must go in the remaining two spots. So each digit combination can form three different numbers. Since there are four digit combinations, there are $4 \times 3 = 12$ cards in the deck that do not contain a 0 but contain two digits the same and whose digits add to 15.

- **Case (d):** The three digits on the card are the same.
  From the earlier cases we discovered only one such combination: $5, 5, 5$. Only one card can be produced using the numbers from this combination.

Combining the counts from the above four cases, there are $8 + 48 + 12 + 1 = 69$ cards in the deck with a digit sum of 15. Therefore, the probability that Joanna selects card whose digits add to 15 is $\frac{69}{900} = \frac{23}{300}$. This translates to approximately a 7.7% chance.

A game is considered *fair* if there is close to a 50% chance of winning. If Joanna was playing a game where she can win by drawing a card whose digits sum to 15, then this game is definitely not fair.

Can you create a game using this specific deck of cards that is reasonably fair and fun to play?

Problem of the Week
Problem E
Nested Squares

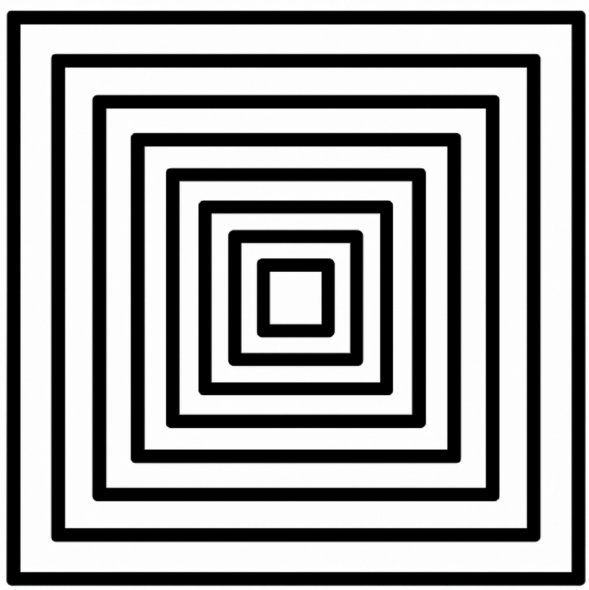
The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 2025^{2025} are perfect squares?



Problem of the Week
Problem E and Solution
Nested Squares

Problem

The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 2025^{2025} are perfect squares?

Solution

First, let's look at the prime factorization of four different perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \ 129\,600 = 2^6 \times 3^4 \times 5^2$$

Note that, in each case, the exponent on each of the prime factors is even. In fact, a positive integer is a perfect square exactly when the exponent on each of the prime factors in its prime factorization is an even integer greater than or equal to zero. Now

$$\begin{aligned} 2025^{2025} &= (3^4 \times 5^2)^{2025} \\ &= (3^4)^{2025} \times (5^2)^{2025} \\ &= 3^{8100} \times 5^{4050} \end{aligned}$$

All positive divisors of 2025^{2025} will be of the form $3^k \times 5^n$, $0 \leq k \leq 8100$, and $0 \leq n \leq 4050$, where k and n are each integers.

For $3^k \times 5^n$ to be a perfect square, k must be an even integer such that $0 \leq k \leq 8100$ and n must be an even integer such that $0 \leq n \leq 4050$.

There are $8100 \div 2 = 4050$ even integers from 1 to 8100. The number 0 is also even, so there are 4051 possible values of k.

There are 2025 even integers from 1 to 4050. The number 0 is also even, so there are 2026 possible values of n.

For each of the 4051 values of k, there are 2026 values of n so there are $4051 \times 2026 = 8\,207\,326$ perfect square divisors of 2025^{2025}.

Therefore, 2025^{2025} has $8\,207\,326$ positive divisors that are perfect squares. This is over 8 million perfect square divisors!

Problem of the Week

Problem E

The Spiral Sequence Challenge

A spiral of the positive integers, placed into rows and columns, is created in the following way:

The integer 1 is placed.

Moving up one row, the integer 2 is placed.

Moving right one column, the integer 3 is placed.

Moving down one row, the integer 4 is placed, then moving down one more row, the integer 5 is placed.

Moving left one column, the integer 6 is placed, then moving left one more column, the integer 7 is placed.

Moving up one row, the integer 8 is placed, then moving up one more row, the integer 9 is placed, then moving up one more row, the integer 10 is placed.

Moving right one column, the integer 11 is placed, then moving right one more column the integer 12 is placed, then moving right one more column, the integer 13 is placed.

The pattern continues, alternating moving down across rows, then left across columns, then up across rows, then right across columns to create a spiral shape.

$$
\begin{array}{ccccccccc}
 & & 10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 \\
 & & \uparrow & & & & & & \downarrow \\
 & & 9 & & 2 & \rightarrow & 3 & & 14 \\
 & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
\vdots & & 8 & & 1 & & 4 & & 15 \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
22 & & 7 & \leftarrow & 6 & \leftarrow & 5 & & 16 \\
\uparrow & & & & & & & & \downarrow \\
21 & \leftarrow & 20 & \leftarrow & 19 & \leftarrow & 18 & \leftarrow & 17
\end{array}
$$

Where do the integers 2024, 2025, and 2026 appear in the spiral?

Problem of the Week

Problem E and Solution

The Spiral Sequence Challenge

Problem

A spiral of the positive integers, placed into rows and columns, is created in the following way:
The integer 1 is placed.
Moving up one row, the integer 2 is placed.
Moving right one column, the integer 3 is placed.
Moving down one row, the integer 4 is placed, then moving down one more row, the integer 5 is placed.
Moving left one column, the integer 6 is placed, then moving left one more column, the integer 7 is placed.
Moving up one row, the integer 8 is placed, then moving up one more row, the integer 9 is placed, then moving up one more row, the integer 10 is placed.
Moving right one column, the integer 11 is placed, then moving right one more column the integer 12 is placed, then moving right one more column, the integer 13 is placed.
The pattern continues, alternating moving down across rows, then left across columns, then up across rows, then right across columns to create a spiral shape.

$$\begin{array}{ccccccccc}
 & & 10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 \\
 & & \uparrow & & & & & & \downarrow \\
 & & 9 & & 2 & \rightarrow & 3 & & 14 \\
 & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
\vdots & & 8 & & 1 & & 4 & & 15 \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
22 & & 7 & \leftarrow & 6 & \leftarrow & 5 & & 16 \\
\uparrow & & & & & & & & \downarrow \\
21 & \leftarrow & 20 & \leftarrow & 19 & \leftarrow & 18 & \leftarrow & 17
\end{array}$$

Where do the integers 2024, 2025, and 2026 appear in the spiral?

Solution

Consider the integers 1, 2, 3, and 4 in the spiral. These integers form a 2 integer by 2 integer "square". This square has 1 in the bottom-left corner, 2 in the top-left corner, 3 in the top-right corner, and 4 in the bottom-right corner.

$$\begin{array}{ccc}
2 & \rightarrow & 3 \\
\uparrow & & \downarrow \\
1 & & 4
\end{array}$$

When we consider the integers 1 through 9, we note that they form a second "square". This square is 3 integers by 3 integers, and has the integer 9 in the top-left corner.

$$\begin{array}{ccccc}
9 & & 2 & \rightarrow & 3 \\
\uparrow & & \uparrow & & \downarrow \\
8 & & 1 & & 4 \\
\uparrow & & & & \downarrow \\
7 & \leftarrow & 6 & \leftarrow & 5
\end{array}$$

When we consider the integers 1 through 16, we note that they form a square that is 4 integers by 4 integers, and has the integer 16 in the bottom-right corner.

$$\begin{array}{ccccccc}
10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 \\
\uparrow & & & & & & \downarrow \\
9 & & 2 & \rightarrow & 3 & & 14 \\
\uparrow & & \uparrow & & \downarrow & & \downarrow \\
8 & & 1 & & 4 & & 15 \\
\uparrow & & & & \downarrow & & \downarrow \\
7 & \leftarrow & 6 & \leftarrow & 5 & & 16
\end{array}$$

Continuing in this manner, the integers 1 through 25 form a square that is 5 integers by 5 integers, and has the integer 25 in the top-left corner.

$$\begin{array}{ccccccccc}
25 & & 10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 \\
\uparrow & & \uparrow & & & & & & \downarrow \\
24 & & 9 & & 2 & \rightarrow & 3 & & 14 \\
\uparrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
23 & & 8 & & 1 & & 4 & & 15 \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
22 & & 7 & \leftarrow & 6 & \leftarrow & 5 & & 16 \\
\uparrow & & & & & & & & \downarrow \\
21 & \leftarrow & 20 & \leftarrow & 19 & \leftarrow & 18 & \leftarrow & 17
\end{array}$$

This pattern continues, and the next square will have $6 \times 6 = 36$ in the bottom-right corner. The following square will have $7 \times 7 = 49$ in the top-left corner.

In general, the integers from 1 to n^2 will form a square with n rows of integers and n columns of integers. Since each new square is completed by alternating creating a row from right to left and then a column from bottom to top, with creating a row from left to right and then a column from top to bottom, all odd perfect squares are in the top-left corners and all even perfect squares are in the bottom-right corners.

Now, $45^2 = 2025$, so 2025 is an odd perfect square.

Thus, 2025 will be the top-left corner of the 45 integer by 45 integer "square". The integer 2024 will appear below 2025 and the integer 2026 will appear above 2025.

Problem of the Week

Problem E

Luminous Detail

A *pixel* is the smallest unit of a digital image.

The number of pixels/cm in each of the horizontal and vertical directions of a digital image affects the quality of the image. The more pixels/cm, the sharper the image will be.

A small monitor has dimensions 15 cm by 10 cm and has 80 pixels/cm in each dimension. The total number of pixels is $(15 \times 80) \times (10 \times 80) = 960\,000$.

The manufacturer wants to build a new monitor with $2\,145\,624$ pixels. To accomplish this, both the length and width of the screen will be increased by $n\%$ and the number of pixels/cm in each dimension will be increased by $2n\%$. Determine the dimensions of the new monitor and the new number of pixels/cm.

Problem of the Week
Problem E and Solution
Luminous Detail

Problem

A *pixel* is the smallest unit of a digital image.

The number of pixels/cm in each of the horizontal and vertical directions of a digital image affects the quality of the image. The more pixels/cm, the sharper the image will be.

A small monitor has dimensions 15 cm by 10 cm and has 80 pixels/cm in each dimension. The total number of pixels is $(15 \times 80) \times (10 \times 80) = 960\,000$.

The manufacturer wants to build a new monitor with 2 145 624 pixels. To accomplish this, both the length and width of the screen will be increased by $n\%$ and the number of pixels/cm in each dimension will be increased by $2n\%$. Determine the dimensions of the new monitor and the new number of pixels/cm.

Solution

Let $a > 0$ represent the percentage increase, expressed as a decimal, in each dimension of the monitor. Then $2a$ represents the percentage increase, expressed as a decimal, in the number of pixels/cm. (So $n = 100a$ and $2n = 200a$.)

We know that

[(New Length)×(New pixels/cm)]×[(New Width)×(New pixels/cm)] = Total Number of Pixels

Thus,

$$\begin{aligned}
[15(1+a) \times 80(1+2a)] \times [10(1+a) \times 80(1+2a)] &= 2\,145\,624 \\
(15)(80)(10)(80)(1+a)^2(1+2a)^2 &= 2\,145\,624 \\
960\,000(1+a)^2(1+2a)^2 &= 2\,145\,624 \\
(1+a)^2(1+2a)^2 &= 2.235\,025
\end{aligned}$$

Taking the square root of both sides, we have $(1+a)(1+2a) = \pm 1.495$.

Since $a > 0$, we have $(1+a)(1+2a) > 0$. Thus, $(1+a)(1+2a) = 1.495$.

Expanding and simplifying, we have $2a^2 + 3a + 1 = 1.495$, and so $2a^2 + 3a - 0.495 = 0$.

Using the quadratic formula, $a = \dfrac{-3 \pm \sqrt{9 - 4(2)(-0.495)}}{4} = \dfrac{-3 \pm \sqrt{12.96}}{4} = \dfrac{-3 \pm 3.6}{4}$.

It follows that $a = 0.15$ or $a = -1.65$. Since we are looking for a percentage increase, then we must have $a > 0$ and so $a = -1.65$ is inadmissible.

Therefore, the dimensions of the screen must each be increased by 15% and the numbers of pixels/cm must be increased by $2 \times 15\% = 30\%$.

The increased length is $15 \times 1.15 = 17.25$ cm and the increased width is $10 \times 1.15 = 11.5$ cm. The increased number of pixels/cm is $80 \times 1.3 = 104$ pixels/cm.

Problem of the Week

Problem E

Just Like Kayak 3

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine the number of nine-digit palindromic integers that satisfy the following:

- The first five digits in the integer are distinct.
- The integer is divisible by 4 and/or the integer ends with an 8.

Problem of the Week
Problem E and Solution
Just Like Kayak 3

Problem

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine the number of nine-digit palindromic integers that satisfy the following:

- The first five digits in the integer are distinct.
- The integer is divisible by 4 and/or the integer ends with an 8.

Solution

Any nine-digit palindromic integer with the first five digits distinct will be of the form $abcdedcba$, where a, b, c, d, and e are distinct digits.

First we will count the number of such palindromes that end with 8. These will be integers of the form $8bcdedcb8$ where $b, c, d, e \neq 8$. In this case, there will be 9 choices for the digit b, 8 choices for the digit c, 7 choices for the digit d, and 6 choices for the digit e. Therefore, the number of palindromes with the first five digits distinct that end with 8 is $9 \times 8 \times 7 \times 6 = 3024$.

Next we will count the number of palindromes with the first five digits distinct that are divisible by 4 but do not end with 8. (We must exclude the palindromes that end with 8 because we already counted them above.) In general, if an integer is divisible by 4 then the integer formed by its last two digits must be divisible by 4. Therefore, if an integer of the form $abcdedcba$ is divisible by 4, then the two-digit integer ba must be divisible by 4. However we cannot have $a = 0$ since the leading digit in our palindrome cannot be 0. Thus, we need to determine the number of two-digit integers ba with $b \neq a$ and $a \neq 0, 8$ that are divisible by 4. These are:

$$04,\ 12,\ 16,\ 24,\ 32,\ 36,\ 52,\ 56,\ 64,\ 72,\ 76,\ 84,\ 92,\ 96$$

In total there are 14 possibilities for ba. For each of these possibilities, there are 8 choices for the digit c, 7 choices for the digit d, and 6 choices for the digit e. Therefore, the number of palindromes with the first five digits distinct that are divisible by 4 but do not end with 8 is $14 \times 8 \times 7 \times 6 = 4704$.

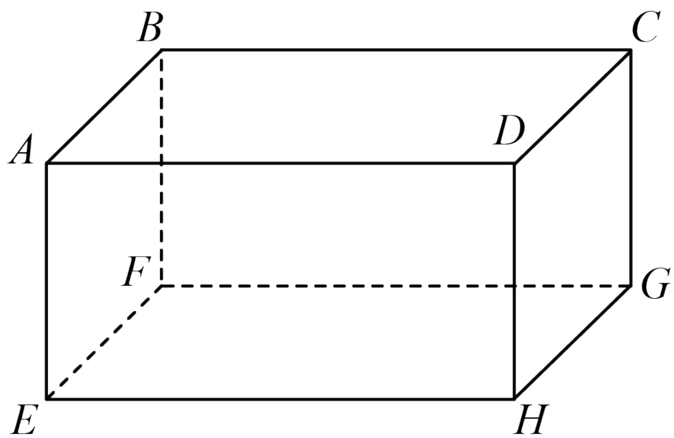
Thus, the number of nine-digit palindromic integers with the first five digits distinct that are divisible by 4 and/or end with 8 is $3024 + 4704 = 7728$.

Problem of the Week

Problem E

Corner Connection

In rectangular prism $ABCDEFGH$, the sum of the lengths of all of the edges is 28 cm, and the total surface area is 13 cm^2.



What is the length of EC, the diagonal of the prism?

Problem of the Week
Problem E and Solution
Corner Connection

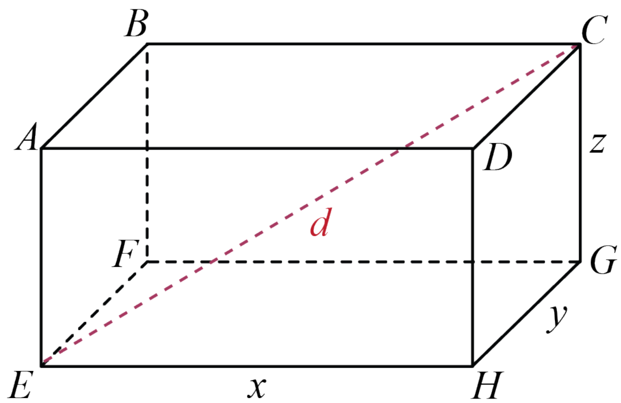
Problem

In rectangular prism $ABCDEFGH$, the sum of the lengths of all of the edges is 28 cm, and the total surface area is 13 cm^2.

What is the length of EC, the diagonal of the prism?

Solution

Let $EH = x$, $HG = y$, and $CG = z$. We construct EC and label it d.



By the Pythagorean Theorem in $\triangle EHG$, $EG^2 = EH^2 + HG^2$.
By the Pythagorean Theorem in $\triangle EGC$, $EC^2 = EG^2 + CG^2$.
Therefore, $EC^2 = EG^2 + CG^2 = EH^2 + HG^2 + CG^2$.
That is, $d^2 = x^2 + y^2 + z^2$.

Since the sum of the lengths of all the edges is 28, then $4x + 4y + 4z = 28$ or $x + y + z = 7$.
Since the surface area of the prism is 13, we know $2xy + 2yz + 2xz = 13$.

Since we have squared terms and pair factor terms it might be helpful to expand $(x + y + z)^2$.

$$\begin{aligned}(x+y+z)^2 &= (x+(y+z))^2\\ &= x^2 + 2x(y+z) + (y+z)^2\\ &= x^2 + 2xy + 2xz + y^2 + 2xy + z^2\\ &= (x^2 + y^2 + z^2) + (2xy + 2xz + 2yz)\end{aligned}$$

Since $x + y + z = 7$, $d^2 = x^2 + y^2 + z^2$, and $2xy + 2yz + 2xz = 13$, we have

$$\begin{aligned}7^2 &= d^2 + 13\\ d^2 &= 49 - 13 = 36\end{aligned}$$

Since $d > 0$, we have $d = 6$. Therefore, the length of EC is 6 cm.

Problem of the Week
Problem E
Favourite Number 3

Matej told his friend Clare the following properties about his favourite number:

- It is a positive seven-digit integer.
- It contains each of the digits from 1 through 7 exactly once.
- The first digit is not 1.
- The second digit is not 2.

Clare then wrote down a number satisfying all these properties. What is the probability that the number Clare wrote was Matej's favourite number?

Problem of the Week
Problem E and Solution
Favourite Number 3

Problem

Matej told his friend Clare the following properties about his favourite number:

- It is a positive seven-digit integer.
- It contains each of the digits from 1 through 7 exactly once.
- The first digit is not 1.
- The second digit is not 2.

Clare then wrote down a number satisfying all these properties. What is the probability that the number Clare wrote was Matej's favourite number?

Solution

Let $abcdefg$ represent any seven-digit number where a, b, c, d, e, f, and g are different digits from 1 through 7.

Note that this solution uses *factorial notation*, as in $4 \times 3 \times 2 \times 1 = 4!$. In general, if n is a positive integer, then $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. We now proceed with two different approaches to solving the problem.

Solution 1

In this solution we determine the number of possible numbers that Clare could have written down directly. Thus, Clare's number is of the form $abcdefg$ where $a \neq 1$ and $b \neq 2$. We consider the following two cases:

- **Case 1:** $a = 2$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 2$.

- **Case 2:** $a \neq 2$
  In this case, there are 5 choices for the digit a (since we also have $a \neq 1$). Once digit a is chosen, there are 5 choices for the digit b (since we have $b \neq 2$ and $b \neq a$), 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 25 \times 5!$ possibilities when $a \neq 2$.

Thus, the total number of possible numbers Claire could have written down is $6! + 25 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.

Solution 2

In this solution we determine the number of possible numbers that Clare could have written down indirectly.

First, we determine the total number of numbers of the form $abcdefg$ where a, b, c, d, e, f, and g are different digits from 1 through 7. There are 7 choices for the digit a, 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$ possibilities.

Next, we determine the number of possible seven-digit numbers of the form $abcdefg$ where $a = 1$ and/or $b = 2$. We consider the following two cases:

- **Case 1:** $a = 1$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 1$.

- **Case 2:** $a \neq 1$ and $b = 2$
  In this case, there are 5 choices for the digit a, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 5!$ possibilities when $a \neq 1$ and $b = 2$.

Thus, the total number of possible numbers Claire could have written down is $7! - 6! - 5 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.

Problem of the Week

Problem E

Cash Divide

A group of friends won a cash award. All of the money was to be distributed among the group members in the following way:

(i) $\$x$ to the oldest member of the group plus $\frac{1}{16}$ of what remains, then

(ii) $\$2x$ to the second oldest member of the group plus $\frac{1}{16}$ of what then remains, then

(iii) $\$3x$ to the third oldest member of the group plus $\frac{1}{16}$ of what then remains, and so on.

When the distribution of the money was complete, each group member received the same amount and no money was left over. Determine the number of members in the group of friends.

Problem of the Week

Problem E and Solution

Cash Divide

Problem

A group of friends won a cash award. All of the money was to be distributed among the group members in the following way:

(i) $\$x$ to the oldest member of the group plus $\frac{1}{16}$ of what remains, then

(ii) $\$2x$ to the second oldest member of the group plus $\frac{1}{16}$ of what then remains, then

(iii) $\$3x$ to the third oldest member of the group plus $\frac{1}{16}$ of what then remains, and so on.

When the distribution of the money was complete, each group member received the same amount and no money was left over. Determine the number of members in the group of friends.

Solution

Let T be the total value of the cash award. Since there is a cash award, $T > 0$.

Let y be the amount of money given to each group member. Since each group member receives money, $y > 0$.

Then $\frac{T}{y}$ is the number of group members.

The oldest group member receives x to begin with. There would be $(T - x)$ left at this point. The oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - x)$. Therefore, the oldest group member receives $y = x + \frac{1}{16}(T - x)$.

The second oldest group member receives $2x$ to begin with. There would now be $(T - y - 2x)$ left at this point. This represents the original amount minus the oldest member's full share minus the amount received so far by the second oldest group member. The second oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - y - 2x)$. Therefore, the second oldest group member receives $y = 2x + \frac{1}{16}(T - y - 2x)$.

But each group member receives the same amount. Since $y = x + \frac{1}{16}(T - x)$ and $y = 2x + \frac{1}{16}(T - y - 2x)$ it follows that

$$x + \frac{1}{16}(T - x) = 2x + \frac{1}{16}(T - y - 2x)$$

Multiplying both sides by 16, we have

$$\begin{aligned}
16x + (T - x) &= 32x + (T - y - 2x) \\
16x + T - x &= 32x + T - y - 2x \\
16x - x &= 32x - y - 2x \\
15x &= 30x - y \\
y &= 15x
\end{aligned}$$

Therefore, each group member receives $\$15x$.

Substituting $15x$ for y into the equation $y = x + \frac{1}{16}(T - x)$ we obtain

$$\begin{aligned}
x + \frac{1}{16}(T - x) &= 15x \\
\frac{1}{16}(T - x) &= 14x \\
T - x &= 224x \\
T &= 225x
\end{aligned}$$

Therefore, the total value of the cash award is $\$225x$.

We can now determine the number of group members $\dfrac{T}{y} = \dfrac{225x}{15x} = 15$.

Therefore, there are 15 group members.

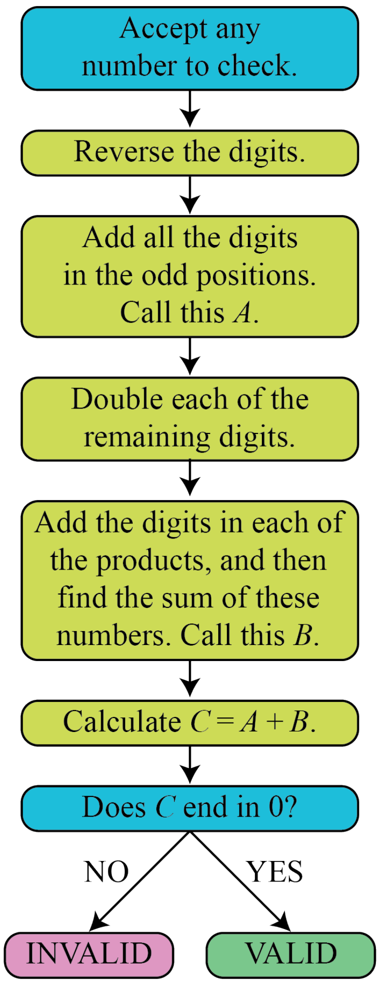
Notice, we did not need to know T, the total value of the cash award. It turns out that once we know x, we can determine each group member's share, $\$15x$, and the total value of the award, $T = \$225x$.

Problem of the Week

Problem E

Check Please 3

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$

- Calculate B:

$$\begin{aligned} B &= (1 + 8) + (1 + 0) + 2 \\ &= 9 + 1 + 2 \\ &= 12 \end{aligned}$$

- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 2643 $XY5X$ 081 is a valid card number when verified by the Luhn Algorithm, where X and Y are single digits and $X \leq Y$. Determine all possible values of X and Y.

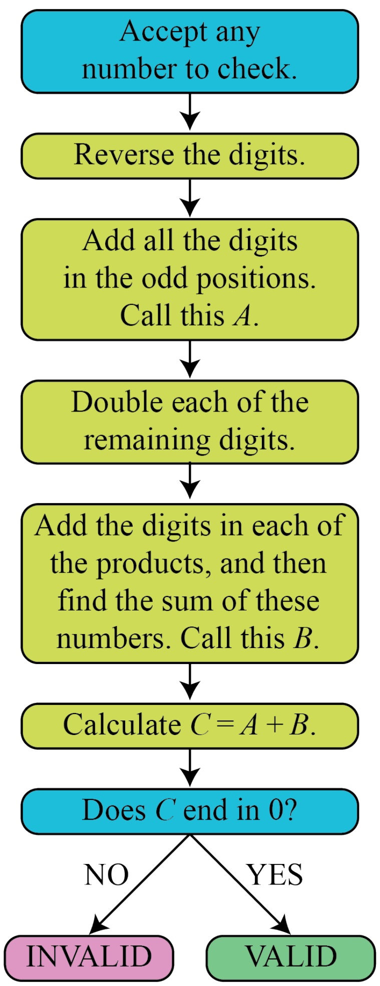
Problem of the Week

Problem E and Solution

Check Please 3

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
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- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

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$$2 \times 1 = 2$$

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- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 2643 $XY5X$ 081 is a valid card number when verified by the Luhn Algorithm, where X and Y are single digits and $X \leq Y$. Determine all possible values of X and Y.

Solution

When the digits of the number are reversed the resulting number is $180\ X5YX\ 3462$. The sum of the digits in the odd positions is $A = 1 + 0 + 5 + X + 4 + 2 = 12 + X$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$2 \times 8 = 16;\ 2 \times X = 2X;\ 2 \times Y = 2Y;\ 2 \times 3 = 6;\ 2 \times 6 = 12$$

Let m represent the sum of the digits of $2X$ and n represent the sum of the digits of $2Y$. When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + m + n + 6 + (1 + 2) = 16 + m + n$$

C is the sum of A and B, so $C = 12 + X + 16 + m + n = 28 + X + m + n$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what possible sums can be obtained? We summarize these in the following table.

Original Digit (X or Y)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2X$ or $2Y$)	0	2	4	6	8	10	12	14	16	18
Sum of the Digits of $2X$ or $2Y$ (m or n)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of $2X$ or $2Y$ can be any integer from 0 to 9, inclusive. It follows that the only values for m or n are the integers from 0 to 9.

Since the number is valid, $C = 28 + X + m + n$ must end in zero.

What are the possible values to consider for C? Since X, m, and n are each integers from 0 to 9, the minimum value of $X + m + n$ is 0, and the maximum is 27. Thus, the minimum value for C is 28 and the maximum is $28 + 27 = 55$. It follows that the only valid possibilities for C that end in zero are 30, 40, and 50. We will consider each of the three possibilities.

- Case 1: $C = 30$.
  Then $X + m + n = 2$. The only possibility is $X = m = 0$ and $n = 2$. Then $Y = 1$. Thus, one solution is $(X, Y) = (0, 1)$.

- Case 2: $C = 40$.
  Then $X + m + n = 12$. We summarize the possibilities in the following table.

X	m	$n = 12 - X - m$	Y	Valid?
0	0	12	not possible, $n > 9$	no, $n > 9$
1	2	9	9	yes, $X \leq Y$
2	4	6	3	yes, $X \leq Y$
3	6	3	6	yes, $X \leq Y$
4	8	0	0	no, $X > Y$
5	1	6	3	no, $X > Y$
6	3	3	6	yes, $X \leq Y$
7	5	0	0	no, $X > Y$
8	7	-3	not possible, $n < 0$	no, $n < 0$
9	9	-6	not possible, $n < 0$	no, $n < 0$

Therefore, this case produces four valid possibilities for (X, Y): $(1, 9)$, $(2, 3)$, $(3, 6)$ and $(6, 6)$.

- Case 3: $C = 50$.
  Then $X + m + n = 22$. We summarize the possibilities in the following table.

X	m	$n = 22 - X - m$	Y	Valid?
0	0	22	not possible, $n > 9$	no, $n > 9$
1	2	19	not possible, $n > 9$	no, $n > 9$
2	4	16	not possible, $n > 9$	no, $n > 9$
3	6	13	not possible, $n > 9$	no, $n > 9$
4	8	10	not possible, $n > 9$	no, $n > 9$
5	1	16	not possible, $n > 9$	no, $n > 9$
6	3	13	not possible, $n > 9$	no, $n > 9$
7	5	10	not possible, $n > 9$	no, $n > 9$
8	7	7	8	yes, $X \leq Y$
9	9	4	2	no, $X > Y$

Therefore, this case produces one valid possibility for (X, Y): $(8, 8)$.

We have examined all possible values for C. Therefore, there are six valid possibilities for (X, Y): $(0, 1)$, $(1, 9)$, $(2, 3)$, $(3, 6)$, $(6, 6)$, and $(8, 8)$.

These correspond to the following six card numbers, which are indeed valid by the Luhn Algorithm:
2643 0150 081, 2643 1951 081, 2643 2352 081, 2643 3653 081, 2643 6656 081, and 2643 8858 081.

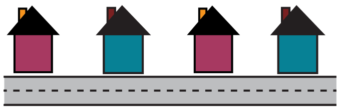
Problem of the Week

Problem E

A Street Called Square

Alyssa, Bilal, Constantine, and Daiyu each live in a house that is numbered with a positive integer.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. Constantine's house number is 31 more than Alyssa's house number, and Daiyu's house number is 31 more than Bilal's house number. What is each person's house number?



Problem of the Week
Problem E and Solution
A Street Called Square

Problem

Alyssa, Bilal, Constantine, and Daiyu each live in a house that is numbered with a positive integer.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. Constantine's house number is 31 more than Alyssa's house number, and Daiyu's house number is 31 more than Bilal's house number. What is each person's house number?

Solution

Both Alyssa's house number and Bilal's house number must be two-digit integers. If Alyssa's house number is a one-digit integer, Bilal's house number would have to be a three-digit integer to create the four-digit perfect square. This means that Constantine's house number would be a two-digit integer and Daiyu's house number would be at least a three-digit integer. Writing Constantine's house number followed by Daiyu's house number would create a five-digit integer. A similar argument can be presented if Bilal's house number is a one-digit integer. Therefore, both Alyssa and Bilal have house numbers that are two-digit integers.

Let Alyssa's house number be x and Bilal's house number be y. Then $100x + y$ is the four-digit integer created by writing Alyssa's house number followed by Bilal's house number. It is given that $100x + y$ is a perfect square, so $100x + y = k^2$ for some positive integer k.

Therefore, Constantine's house number is $(x + 31)$ and Daiyu's house number is $(y + 31)$. The number created by writing Constantine's house number followed by Daiyu's house number is $100(x + 31) + (y + 31)$. This four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer m. Simplifying, we have

$$\begin{aligned} 100x + 3100 + y + 31 &= m^2 \\ 100x + y + 3131 &= m^2 \end{aligned}$$

Substituting $100x + y = k^2$, we obtain $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$.

Since $m^2 - k^2$ is a difference of squares, we have $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m+k$ is a positive integer and $m+k > m-k$. Also, $m-k$ must be a positive integer since $(m+k)(m-k) = 3131$. So we are looking for two positive integers that multiply to 3131. There are two possibilities, 3131×1 or 101×31.

First we will examine $(m+k)(m-k) = 3131 \times 1$. From this we obtain two equations in two unknowns, namely $m+k = 3131$ and $m-k = 1$. Subtracting the second equation from the first gives $2k = 3130$ or $k = 1565$. Then $k^2 = 1565^2 = 2\,449\,225$. This is not a four-digit number, so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m+k)(m-k) = 101 \times 31$. This leads to $m+k = 101$ and $m-k = 31$. Subtracting the second equation from the first, we get $2k = 70$ or $k = 35$. Then $100x + y = k^2 = 1225$. Therefore, $x = 12$ and $y = 25$, since 1225 is the four-digit number formed by writing Alyssa's house number, x, followed by Bilal's house number, y.

Now, Constantine's house number is $x + 31 = 12 + 31 = 43$ and Daiyu's house number is $y + 31 = 25 + 31 = 56$. Notice that $4356 = 66^2$, so it is a four-digit perfect square.

Therefore, Alyssa's house number is 12, Bilal's house number is 25, Constantine's house number is 43, and Daiyu's house number is 56.

Problem of the Week

Problem E

Stock the Shelves

A local food bank has created a unique 100-day plan for collecting canned food donations.

Day 1 Goal: Collect 50 cans of food.

Day 2 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 1.

Day 3 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 2.

Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

$\vdots$

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100^{th} day?

Note:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Problem of the Week

Problem E and Solution

Stock the Shelves

Problem

A local food bank has created a unique 100-day plan for collecting canned food donations.

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Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

$\vdots$

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100^{th} day?

Solution

We will introduce function notation to represent the information in the problem.

Let n represent the day number and $f(n)$ represent the number of cans collected on day n.

We know that on the first day, 50 cans were collected. So, $f(1) = 50$.

On the second day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 1. So, $f(2) = 3 + 2 + f(1) = 3 + 2 + 50 = 55$.

On the third day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 2. So, $f(3) = 3 + 3 + f(2) = 3 + 3 + 55 = 61$.

On the fourth day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 3. So, $f(4) = 3 + 4 + f(3) = 3 + 4 + 61 = 68$.

On the n^{th} day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day $(n - 1)$. So, $f(n) = 3 + n + f(n - 1)$.

This sequence of cans collected can be defined recursively as follows:

$$f(n) = \begin{cases} 50, & \text{if } n = 1 \\ f(n-1) + n + 3, & \text{if } n \geq 2 \text{ and } n \in \mathbb{Z} \end{cases}$$

We want to find $f(100)$, the 100^{th} term in the sequence.

Solution 1

If $f(n-1)$ and $f(n)$, with $n \geq 2$, are adjacent terms in the sequence of cans collected, then $f(n) = f(n-1) + n + 3$. Rearranging, $f(n) - f(n-1) = n + 3$. That is, the difference between consecutive terms will be the term number of the term in the higher position plus 3.

Thus, $f(100)$ will be equal to $f(1) = 50$, plus the sum of the term numbers from 2 to 100, plus 3 a total of 99 times. That is,

$$\begin{aligned} f(100) &= 50 + (2 + 3 + 4 + \cdots + 100) + 3(99) \\ &= 50 + (1 + 2 + 3 + \cdots + 100) - 1 + 3(99) \\ &= 50 + \frac{100(101)}{2} - 1 + 3(99) \\ &= 50 + 5050 - 1 + 297 \\ &= 5396 \end{aligned}$$

Therefore, 5396 cans are collected on the 100^{th} day.

Solution 2

We summarize the number of cans collected on the first 6 days in a table:

Day Number	1	2	3	4	5	6
Number of Cans Collected	50	55	61	68	76	85

If we treat the day number as the independent variable, say x, and the number of cans collected as the dependent variable, say y, we can make some conclusions. First, Δx is constant. However, Δy, the first differences, are not constant. Therefore, the function is not linear.

The first differences of the given values are 5, 6, 7, 8, and 9. If we calculate the second differences, each is 1. That is, for a constant Δx, the first differences are not constant but the second differences are constant. We will leave it to the reader to justify that this pattern is true for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $x \geq 1, x \in \mathbb{Z}$.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 50$, so

$$a + b + c = 50 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 55$, so

$$4a + 2b + c = 55 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 61$, so

$$9a + 3b + c = 61 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 5$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 6$, which we call equation (5).

We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 5$, and so $b = 5 - \frac{3}{2} = \frac{7}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{7}{2}$ into equation (1), we have $\frac{1}{2} + \frac{7}{2} + c = 50$, and so $c = 50 - 4 = 46$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{7}{2}x + 46$, where $x \geq 1, x \in \mathbb{Z}$.

To determine the value collected on the 100$^{\text{th}}$ day, we evaluate $f(x)$ at $x = 100$.

$$f(100) = \frac{1}{2}(100)^2 + \frac{7}{2}(100) + 46 = 5000 + 350 + 46 = 5396$$

Therefore, 5396 cans are collected on the 100$^{\text{th}}$ day.

Using this approach, we can find the amount collected on any day, since we have a general formula for the value collected given the day number. That is, the amount collected on day n is $f(n) = \frac{1}{2}n^2 + \frac{7}{2}n + 46$, where $n \geq 1, n \in \mathbb{Z}$.

Solution 3

When $n \geq 2$, we have $f(n) = f(n-1) + n + 3$. Rearranging, we obtain $f(n) - f(n-1) = n + 3$. Using $f(n) - f(n-1) = n + 3$ with integer values of n from 100 to 2, we have

$$\begin{aligned}
f(100) - f(99) &= 100 + 3 \\
f(99) - f(98) &= 99 + 3 \\
f(98) - f(97) &= 98 + 3 \\
&\vdots \\
f(3) - f(2) &= 3 + 3 \\
f(2) - f(1) &= 2 + 3
\end{aligned}$$

If we add all of the terms on the left side of each equal sign and then simplify, we are left with only $f(100) - f(1)$. If we add all of the terms on the right side of each equal sign, we get

$$2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3)$$

Therefore,

$$\begin{aligned}
f(100) - f(1) &= 2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3) \\
&= (1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100) - 1 + 99(3) \\
&= \frac{100(101)}{2} - 1 + 99(3) \\
&= 5050 - 1 + 297 \\
&= 5346
\end{aligned}$$

Since $f(1) = 50$, we have $f(100) - 50 = 5346$, and so $f(100) = 5396$.

That is, 5396 cans are collected on the 100$^{\text{th}}$ day.

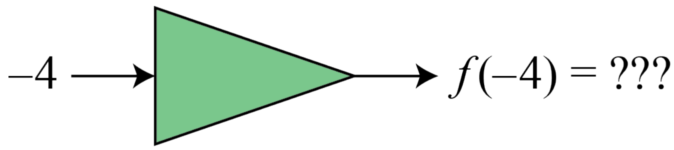
Extension: Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?

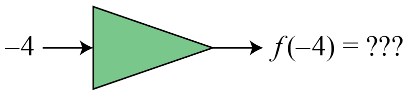
Problem of the Week

Problem E

How Does it Function?

The function $f(x) = x^5 - 4x^4 + ax^3 - x^2 + bx - 8$ has a value of 128 when $x = 4$. Determine the value of the function when $x = -4$.





Problem of the Week

Problem E and Solution

How Does it Function?

Problem

The function $f(x) = x^5 - 4x^4 + ax^3 - x^2 + bx - 8$ has a value of 128 when $x = 4$. Determine the value of the function when $x = -4$.

Solution

We know that the function has a value of 128 when $x = 4$. That is, $f(4) = 128$. Then,

$$\begin{aligned} f(4) &= 128 \\ (4)^5 - 4(4)^4 + a(4)^3 - (4)^2 + b(4) - 8 &= 128 \\ 1024 - 1024 + 64a - 16 + 4b - 8 &= 128 \\ 64a + 4b &= 152 \end{aligned}$$

We want to determine the value of the function when $x = -4$. That is, we want to determine $f(-4)$. Then,

$$\begin{aligned} f(-4) &= (-4)^5 - 4(-4)^4 + a(-4)^3 - (-4)^2 + b(-4) - 8 \\ &= -1024 - 1024 - 64a - 16 - 4b - 8 \\ &= -64a - 4b - 2072 \end{aligned}$$

However, we know from earlier that $64a + 4b = 152$. Then,

$$f(-4) = -(64a + 4b) - 2072 = -152 - 2072 = -2224$$

Therefore, the value of the function is -2224 when $x = -4$.

Note that we are not given enough information to determine the values of a and b, however enough information is given to determine the value of $f(-4)$.

Problem of the Week

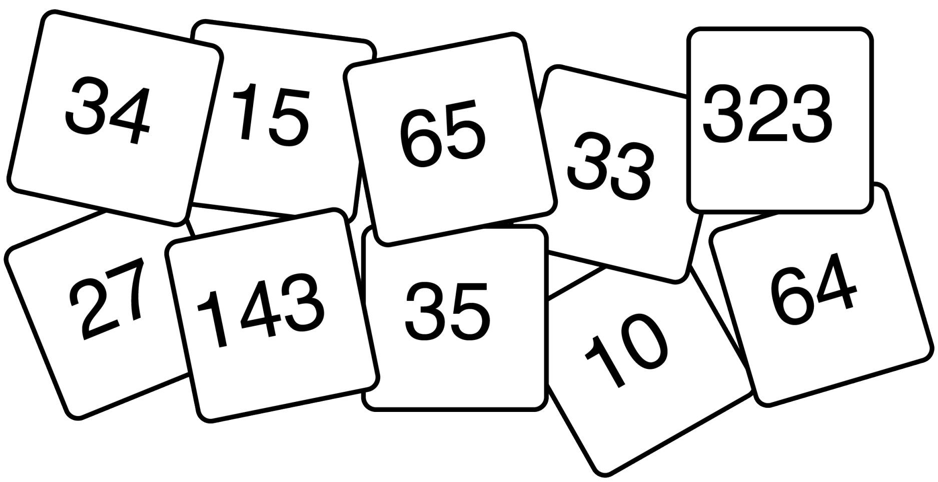
Problem E

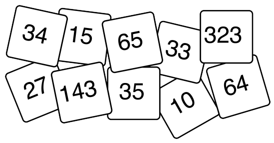
Prime Factor Puzzle

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.





Problem of the Week

Problem E and Solution

Prime Factor Puzzle

Problem

Dani has 10 cards, each with a number on one side. The numbers on the cards are 10, 15, 27, 33, 34, 35, 64, 65, 143, and 323. The cards are placed on a table with the numbers facing up.

Dani takes a card and flips it over so it is now face down. Of the remaining cards that are still face up, the next card she flips over must have a prime factor in common with the card she last flipped over. She continues in this way until either all the cards have been flipped over, or she is unable to flip any of the cards that remain face up.

List all the possible orders in which Dani can flip the cards so that all cards get flipped over.

Solution

We will start by writing down the prime factors for each of the numbers on the cards.

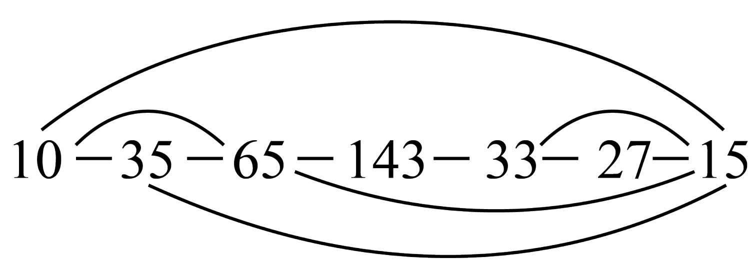
Number	Prime Factors
10	2, 5
15	3, 5
27	3
33	3, 11
34	2, 17

Number	Prime Factors
35	5, 7
64	2
65	5, 13
143	11, 13
323	17, 19

Notice the number 323 shares a prime factor with only the number 34. That means we must start (or end) with 323.

If we start with 323, then the next number must be 34. From 34, the next number could be 64 or 10. If the next number were 10, then in order to eventually flip over the 64 card, the 64 must follow the 10, and at this point no more cards can be flipped. Therefore, in order to flip all the cards, the first four numbers flipped must be 323, 34, 64, and then 10.

From this point, we can draw lines between numbers that share prime factors to create the following diagram.



We now need to figure out the number of paths through the diagram, starting at 10, that use each number exactly once.

After 10, the next number can be 35, 65, or 15. In each case, we carefully trace through all possible paths.

Case 1: The number 35 follows the number 10. That gives us the following five possible paths:

35, 15, 65, 143, 33, 27
35, 15, 27, 33, 143, 65
35, 65, 15, 27, 33, 143
35, 65, 143, 33, 27, 15
35, 65, 143, 33, 15, 27

Case 2: The number 65 follows the number 10. That gives us the following two possible paths:

65, 143, 33, 27, 15, 35
65, 35, 15, 27, 33, 143

Case 3: The number 15 follows the number 10. That gives us the following two possible paths:

15, 27, 33, 143, 65, 35
15, 35, 65, 143, 33, 27

Therefore, starting with the card numbered 323, we have found that there is a total of nine orders for flipping the cards:

323, 34, 64, 10, 35, 15, 65, 143, 33, 27
323, 34, 64, 10, 35, 15, 27, 33, 143, 65
323, 34, 64, 10, 35, 65, 15, 27, 33, 143
323, 34, 64, 10, 35, 65, 143, 33, 27, 15
323, 34, 64, 10, 35, 65, 143, 33, 15, 27
323, 34, 64, 10, 65, 143, 33, 27, 15, 35
323, 34, 64, 10, 65, 35, 15, 27, 33, 143
323, 34, 64, 10, 15, 27, 33, 143, 65, 35
323, 34, 64, 10, 15, 35, 65, 143, 33, 27

Note that each of these can be reversed. Therefore, there are 18 orders in which Dani can flip the cards so that all cards get flipped over. They are the 9 orders listed above and the reverse of each.