Problem of the Week
Problem E and Solution
Cash Divide

Problem

A group of friends won a cash award. All of the money was to be distributed among the group members in the following way:

(i) $\$x$ to the oldest member of the group plus $\frac{1}{16}$ of what remains, then

(ii) $\$2x$ to the second oldest member of the group plus $\frac{1}{16}$ of what then remains, then

(iii) $\$3x$ to the third oldest member of the group plus $\frac{1}{16}$ of what then remains, and so on.

When the distribution of the money was complete, each group member received the same amount and no money was left over. Determine the number of members in the group of friends.

Solution

Let T be the total value of the cash award. Since there is a cash award, $T > 0$.

Let y be the amount of money given to each group member. Since each group member receives money, $y > 0$.

Then $\frac{T}{y}$ is the number of group members.

The oldest group member receives x to begin with. There would be $(T - x)$ left at this point. The oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - x)$. Therefore, the oldest group member receives $y = x + \frac{1}{16}(T - x)$.

The second oldest group member receives $2x$ to begin with. There would now be $(T - y - 2x)$ left at this point. This represents the original amount minus the oldest member's full share minus the amount received so far by the second oldest group member. The second oldest group member then receives $\frac{1}{16}$ of the remaining amount $(T - y - 2x)$. Therefore, the second oldest group member receives $y = 2x + \frac{1}{16}(T - y - 2x)$.

But each group member receives the same amount. Since $y = x + \frac{1}{16}(T - x)$ and $y = 2x + \frac{1}{16}(T - y - 2x)$ it follows that

$$x + \frac{1}{16}(T - x) = 2x + \frac{1}{16}(T - y - 2x)$$

Multiplying both sides by 16, we have

$$16x + (T - x) = 32x + (T - y - 2x)$$
$$16x + T - x = 32x + T - y - 2x$$
$$16x - x = 32x - y - 2x$$
$$15x = 30x - y$$
$$y = 15x$$

Therefore, each group member receives $\$15x$.

Substituting $15x$ for y into the equation $y = x + \frac{1}{16}(T - x)$ we obtain

$$x + \frac{1}{16}(T - x) = 15x$$
$$\frac{1}{16}(T - x) = 14x$$
$$T - x = 224x$$
$$T = 225x$$

Therefore, the total value of the cash award is $\$225x$.

We can now determine the number of group members $\dfrac{T}{y} = \dfrac{225x}{15x} = 15$.

Therefore, there are 15 group members.

Notice, we did not need to know T, the total value of the cash award. It turns out that once we know x, we can determine each group member's share, $\$15x$, and the total value of the award, $T = \$225x$.