Problem of the Week

Problem E and Solution

Just Like Kayak 3

Problem

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine the number of nine-digit palindromic integers that satisfy the following:

- The first five digits in the integer are distinct.
- The integer is divisible by 4 and/or the integer ends with an 8.

Solution

Any nine-digit palindromic integer with the first five digits distinct will be of the form $abcdedcba$, where a, b, c, d, and e are distinct digits.

First we will count the number of such palindromes that end with 8. These will be integers of the form $8bcdedcb8$ where $b, c, d, e \neq 8$. In this case, there will be 9 choices for the digit b, 8 choices for the digit c, 7 choices for the digit d, and 6 choices for the digit e. Therefore, the number of palindromes with the first five digits distinct that end with 8 is $9 \times 8 \times 7 \times 6 = 3024$.

Next we will count the number of palindromes with the first five digits distinct that are divisible by 4 but do not end with 8. (We must exclude the palindromes that end with 8 because we already counted them above.) In general, if an integer is divisible by 4 then the integer formed by its last two digits must be divisible by 4. Therefore, if an integer of the form $abcdedcba$ is divisible by 4, then the two-digit integer ba must be divisible by 4. However we cannot have $a = 0$ since the leading digit in our palindrome cannot be 0. Thus, we need to determine the number of two-digit integers ba with $b \neq a$ and $a \neq 0, 8$ that are divisible by 4. These are:

$$04,\ 12,\ 16,\ 24,\ 32,\ 36,\ 52,\ 56,\ 64,\ 72,\ 76,\ 84,\ 92,\ 96$$

In total there are 14 possibilities for ba. For each of these possibilities, there are 8 choices for the digit c, 7 choices for the digit d, and 6 choices for the digit e. Therefore, the number of palindromes with the first five digits distinct that are divisible by 4 but do not end with 8 is $14 \times 8 \times 7 \times 6 = 4704$.

Thus, the number of nine-digit palindromic integers with the first five digits distinct that are divisible by 4 and/or end with 8 is $3024 + 4704 = 7728$.