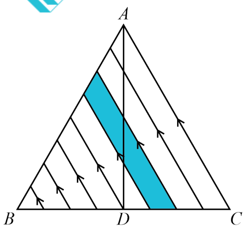


Problem of the Week
Problem E and Solution
Yet Another Suncatcher

Problem

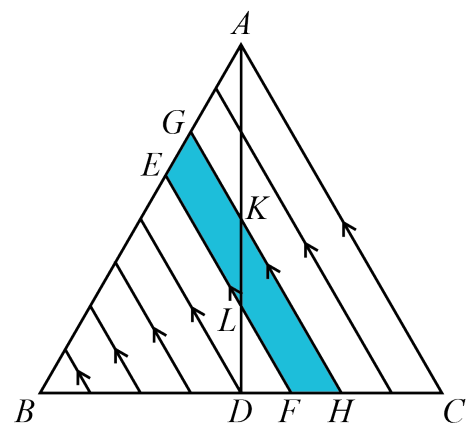
A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

- Sides AB and BC are each divided into 8 segments of equal length.
- Each point of division on AB is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

One of the sections is coloured blue, as shown. An altitude is constructed from A to D on BC, dividing the blue section into two parts. In the blue section, determine the ratio of the area on the left side of the altitude to the area on the right side of the altitude.

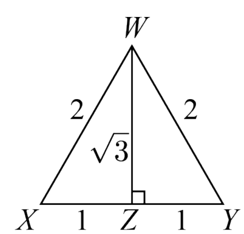
Solution

Solution 1



Since AB and BC are each divided into 8 equal segments, then each segment will have a length of $144 \div 8 = 18$ mm.

Label the blue section $EFHG$. Let K be the point of intersection between altitude AD and segment GH, and L be the point of intersection between altitude AD and segment EF. We want to find the ratio of the area of $GELK$ to the area of $KLFH$.



We first mention a few facts that we will use in our solution. For some equilateral $\triangle WXY$ with side length 2, altitude WZ bisects base XY and it follows that $XZ = ZY = 1$. Using the Pythagorean Theorem, the length of WZ is $\sqrt{2^2 - 1^2} = \sqrt{3}$. In any equilateral triangle, the ratio of the height to the side length is $\sqrt{3} : 2$. In other words, the height of any equilateral triangle is $\frac{\sqrt{3}}{2}$ times its side length. The altitude WZ also bisects $\angle XWY$, so $\angle XWZ = \angle YWZ = 30^\circ$ and $\triangle WXZ$ is a $30^\circ - 60^\circ - 90^\circ$ triangle whose sides are in the ratio $1 : \sqrt{3} : 2$.

Now going back to our problem, we will start by subtracting the area of $\triangle BEF$ from the area of $\triangle BGH$ to find the area of section $EFHG$.

We know that $\angle ABC = \angle BCA = \angle BAC = 60^\circ$. In $\triangle BEF$, $\angle EBF = \angle ABC = 60^\circ$ since they are the same angle. Since $EF \parallel AC$, $\angle BFE = \angle BCA = 60^\circ$ and $\angle BEF = \angle BAC = 60^\circ$. Since all of the angles in $\triangle BEF$ are 60°, it is an equilateral triangle with side length $5 \times 18 = 90$ mm. Using our above result, the height of $\triangle BEF = \frac{\sqrt{3}}{2} \times 90 = 45\sqrt{3}$. The area of $\triangle BEF = \frac{90 \times 45\sqrt{3}}{2} = 2025\sqrt{3}$ mm^2.

In a similar way, we can show that $\triangle BGH$ is equilateral with side length $6 \times 18 = 108$ mm. The height of $\triangle BGH = \frac{\sqrt{3}}{2} \times 108 = 54\sqrt{3}$ and the area of $\triangle BGH = \frac{108 \times 54\sqrt{3}}{2} = 2916\sqrt{3}$ mm^2. Thus,

$$\begin{aligned} \text{area of } EFHG &= \text{area of } \triangle BGH - \text{area of } \triangle BEF \\ &= 2916\sqrt{3} - 2025\sqrt{3} = 891\sqrt{3} \text{ mm}^2 \end{aligned}$$

Next, we find the area of $KLFH$ by finding the area of $\triangle LDF$ and subtracting it from the area of $\triangle KDH$.

In $\triangle LDF$, $\angle LDF = \angle ADC = 90°$, since the altitude AD is perpendicular to base BC and the two angles represent the same angle. Since $EF \parallel AC$, $\angle DFL = \angle BCA = 60°$. It follows that $\angle DLF = 30°$. Therefore, $\triangle LDF$ is a $30° - 60° - 90°$ triangle and $DF : DL : LF = 1 : \sqrt{3} : 2$. Since $DF = 18$, it follows that $DL = 18\sqrt{3}$. The area of $\triangle LDF = \frac{18 \times 18\sqrt{3}}{2} = 162\sqrt{3}$ mm^2.

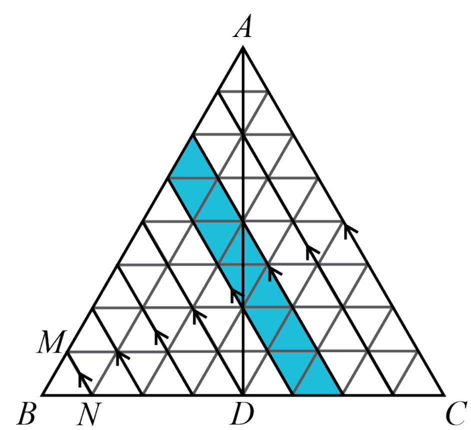
In a similar way, we can show that $\triangle KDH$ is a $30° - 60° - 90°$ triangle with $DH : DK : KH = 1 : \sqrt{3} : 2$. Since $DH = 2 \times 18 = 36$, it follows that $DK = 36\sqrt{3}$. The area of $\triangle KDH = \frac{36 \times 36\sqrt{3}}{2} = 648\sqrt{3}$ mm^2. Then,

$$\text{area of } KLFH = \text{area of } \triangle KDH \ - \text{area of } \triangle LDF = 648\sqrt{3} - 162\sqrt{3} = 486\sqrt{3} \text{ mm}^2$$
$$\text{area of } GELK = \text{area of } EFHG \ - \text{area of } KLFH = 891\sqrt{3} - 486\sqrt{3} = 405\sqrt{3} \text{ mm}^2$$

Therefore, the ratio of the area of $GELK$ to the area of $KLFH$ is $405\sqrt{3} : 486\sqrt{3}$ or $5 : 6$.

Solution 2

Label the endpoints of the line segment closest to B as M and N. Observe that $\triangle ABC$ can be tiled with small equilateral triangles congruent to $\triangle BMN$. That is, equilateral triangles with side length 18 mm. A complete justification of this is not provided here but you may wish to verify this for yourself.



In the second section from B there are 3 small equilateral triangles. In the third section there are 5 small equilateral triangles. In the fourth section there are 7 small equilateral triangles, and so on. The blue section corresponds to the sixth section and contains 11 small equilateral triangles.

Of these 11 small equilateral triangles, there are $4 + \frac{1}{2} + \frac{1}{2} = 5$ on the left of altitude AD. A complete justification of this is not provided here, but you may wish to verify this yourself. Thus, there are $11 - 5 = 6$ on the right of altitude AD. Note that each of these small equilateral triangles have the same area. Therefore, the ratio of the blue area on the left side of the altitude to the blue area on the right side of the altitude is $5 : 6$.