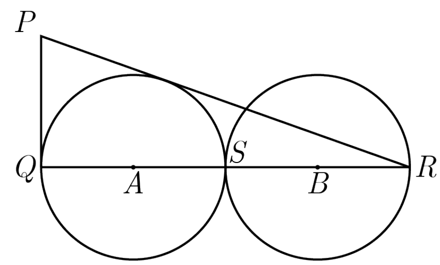
Problem of the Week

Problem E and Solution

A Triangle and Two Circles

Problem

Two circles with centres A and B, each with a radius of 3, are tangent to each other at S. A straight line is drawn through A, S and B, meeting the circle with centre A at Q, $Q \neq S$, and the circle with centre B at R, $R \neq S$. Point P is then drawn so that PQ and PR are each tangent to the circle with centre A.



Determine the length of PQ.

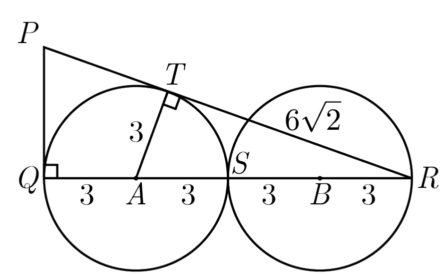
Note: For this problem, you may want to use the following known results about circles:

1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
2. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.

Solution

Since each circle has a radius of 3, $AQ = AS = BS = BR = 3$. From Result 1 given after the problem, since PQ is tangent to the circle with centre A, it follows that $PQ \perp QR$.

Let T be the point of tangency of the line PR. Then $AT \perp PR$. Since AT is a radius of the circle, $AT = 3$. Since $\triangle ATR$ is a right-angled triangle, $TR^2 = AR^2 - AT^2 = 9^2 - 3^2 = 72$. Thus, $TR = \sqrt{72} = 6\sqrt{2}$, since $TR > 0$.



We will now proceed with three different approaches.

Solution 1

In this solution we use trigonometry. Since $PQ \perp QR$ and $AT \perp PR$, $\triangle PQR$ and $\triangle ATR$ are right-angled triangles. In $\triangle PQR$, $\tan R = \frac{PQ}{QR} = \frac{PQ}{12}$. Similarly, in $\triangle ATR$, $\tan R = \frac{AT}{TR} = \frac{3}{6\sqrt{2}}$. Thus,

$$\frac{PQ}{12} = \frac{3}{6\sqrt{2}}$$

$$PQ = \frac{3}{6\sqrt{2}} \times 12 = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Therefore, the length of PQ is $3\sqrt{2}$.

Solution 2

In this solution we use similar triangles. Since $\angle PQR = \angle ATR = 90^\circ$ and $\angle PRQ = \angle ART$, it follows that $\triangle PQR \sim \triangle ATR$. Thus,

$$\frac{PQ}{QR} = \frac{AT}{TR}$$

$$\frac{PQ}{12} = \frac{3}{6\sqrt{2}}$$

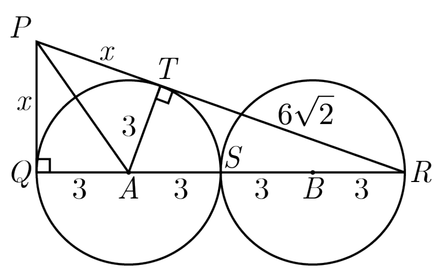
$$PQ = \frac{3}{6\sqrt{2}} \times 12 = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Therefore, the length of PQ is $3\sqrt{2}$.

Notice that while the initial approach is different, this solution is quite similar to Solution 1.

Solution 3

In this solution we use the Pythagorean Theorem. Draw line segment AP. Then $\triangle PQA$ and $\triangle PTA$ are right-angled triangles. Thus, $AP^2 = PQ^2 + AQ^2$ and $AP^2 = PT^2 + AT^2$. Since AQ and AT are both radii, $AQ = AT$. Thus, $PQ = PT$. Let $x = PQ = PT$.



Since $PQ \perp QR$, $\triangle PQR$ is right-angled with $PQ = x$, $QR = 12$, and $PR = x + 6\sqrt{2}$. Thus,

$$PR^2 = PQ^2 + QR^2$$

$$(x + 6\sqrt{2})^2 = x^2 + 12^2$$

$$x^2 + (12\sqrt{2})x + 72 = x^2 + 144$$

$$(12\sqrt{2})x = 72$$

$$x = \frac{72}{12\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Therefore, the length of PQ is $3\sqrt{2}$.