Problem of the Week
Problem E and Solution
Favourite Number 3

Problem

Matej told his friend Clare the following properties about his favourite number:

- It is a positive seven-digit integer.
- It contains each of the digits from 1 through 7 exactly once.
- The first digit is not 1.
- The second digit is not 2.

Clare then wrote down a number satisfying all these properties. What is the probability that the number Clare wrote was Matej's favourite number?

Solution

Let $abcdefg$ represent any seven-digit number where a, b, c, d, e, f, and g are different digits from 1 through 7.

Note that this solution uses *factorial notation*, as in $4 \times 3 \times 2 \times 1 = 4!$. In general, if n is a positive integer, then $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. We now proceed with two different approaches to solving the problem.

Solution 1

In this solution we determine the number of possible numbers that Clare could have written down directly. Thus, Clare's number is of the form $abcdefg$ where $a \neq 1$ and $b \neq 2$. We consider the following two cases:

- **Case 1:** $a = 2$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 2$.

- **Case 2:** $a \neq 2$
  In this case, there are 5 choices for the digit a (since we also have $a \neq 1$). Once digit a is chosen, there are 5 choices for the digit b (since we have $b \neq 2$ and $b \neq a$), 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 25 \times 5!$ possibilities when $a \neq 2$.

Thus, the total number of possible numbers Claire could have written down is $6! + 25 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.

Solution 2

In this solution we determine the number of possible numbers that Clare could have written down indirectly.

First, we determine the total number of numbers of the form $abcdefg$ where a, b, c, d, e, f, and g are different digits from 1 through 7. There are 7 choices for the digit a, 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$ possibilities.

Next, we determine the number of possible seven-digit numbers of the form $abcdefg$ where $a = 1$ and/or $b = 2$. We consider the following two cases:

- **Case 1:** $a = 1$
  In this case, there are 6 choices for the digit b, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$ possibilities when $a = 1$.

- **Case 2:** $a \neq 1$ and $b = 2$
  In this case, there are 5 choices for the digit a, 5 choices for the digit c, 4 choices for the digit d, 3 choices for the digit e, 2 choices for the digit f, and 1 choice for the digit g. Thus, there are $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 5!$ possibilities when $a \neq 1$ and $b = 2$.

Thus, the total number of possible numbers Claire could have written down is $7! - 6! - 5 \times 5! = 3720$. Since only one of these is Matej's favourite number, the probability that the number Clare wrote was Matej's favourite number is $\frac{1}{3720}$.