



Problem of the Week

Problem E and Solution

Stock the Shelves

Problem

A local food bank has created a unique 100-day plan for collecting canned food donations.

Day 1 Goal: Collect 50 cans of food.

Day 2 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 1.

Day 3 Goal: Collect 3 more cans of food than the current day number plus the same number of cans collected on day 2.

Day 4 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 3.

⋮

Day 100 Goal: Collect 3 more cans of food than the day number plus the same number of cans collected on day 99.

How many cans of food will the food bank collect on the 100th day?

Solution

We will introduce function notation to represent the information in the problem.

Let n represent the day number and $f(n)$ represent the number of cans collected on day n .

We know that on the first day, 50 cans were collected. So, $f(1) = 50$.

On the second day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 1. So, $f(2) = 3 + 2 + f(1) = 3 + 2 + 50 = 55$.

On the third day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 2. So, $f(3) = 3 + 3 + f(2) = 3 + 3 + 55 = 61$.

On the fourth day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day 3. So, $f(4) = 3 + 4 + f(3) = 3 + 4 + 61 = 68$.

On the n^{th} day, they collect 3 more cans of food than the current day number plus the same number of cans collected on day $(n - 1)$. So, $f(n) = 3 + n + f(n - 1)$.

This sequence of cans collected can be defined recursively as follows:

$$f(n) = \begin{cases} 50, & \text{if } n = 1 \\ f(n - 1) + n + 3, & \text{if } n \geq 2 \text{ and } n \in \mathbb{Z} \end{cases}$$

We want to find $f(100)$, the 100th term in the sequence.



Solution 1

If $f(n-1)$ and $f(n)$, with $n \geq 2$, are adjacent terms in the sequence of cans collected, then $f(n) = f(n-1) + n + 3$. Rearranging, $f(n) - f(n-1) = n + 3$. That is, the difference between consecutive terms will be the term number of the term in the higher position plus 3.

Thus, $f(100)$ will be equal to $f(1) = 50$, plus the sum of the term numbers from 2 to 100, plus 3 a total of 99 times. That is,

$$\begin{aligned} f(100) &= 50 + (2 + 3 + 4 + \cdots + 100) + 3(99) \\ &= 50 + (1 + 2 + 3 + \cdots + 100) - 1 + 3(99) \\ &= 50 + \frac{100(101)}{2} - 1 + 3(99) \\ &= 50 + 5050 - 1 + 297 \\ &= 5396 \end{aligned}$$

Therefore, 5396 cans are collected on the 100th day.

Solution 2

We summarize the number of cans collected on the first 6 days in a table:

Day Number	1	2	3	4	5	6
Number of Cans Collected	50	55	61	68	76	85

If we treat the day number as the independent variable, say x , and the number of cans collected as the dependent variable, say y , we can make some conclusions. First, Δx is constant. However, Δy , the first differences, are not constant. Therefore, the function is not linear.

The first differences of the given values are 5, 6, 7, 8, and 9. If we calculate the second differences, each is 1. That is, for a constant Δx , the first differences are not constant but the second differences are constant. We will leave it to the reader to justify that this pattern is true for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $x \geq 1, x \in \mathbb{Z}$.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 50$, so

$$a + b + c = 50 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 55$, so

$$4a + 2b + c = 55 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 61$, so

$$9a + 3b + c = 61 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 5$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 6$, which we call equation (5).



We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 5$, and so $b = 5 - \frac{3}{2} = \frac{7}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{7}{2}$ into equation (1), we have $\frac{1}{2} + \frac{7}{2} + c = 50$, and so $c = 50 - 4 = 46$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{7}{2}x + 46$, where $x \geq 1, x \in \mathbb{Z}$.

To determine the value collected on the 100th day, we evaluate $f(x)$ at $x = 100$.

$$f(100) = \frac{1}{2}(100)^2 + \frac{7}{2}(100) + 46 = 5000 + 350 + 46 = 5396$$

Therefore, 5396 cans are collected on the 100th day.

Using this approach, we can find the amount collected on any day, since we have a general formula for the value collected given the day number. That is, the amount collected on day n is $f(n) = \frac{1}{2}n^2 + \frac{7}{2}n + 46$, where $n \geq 1, n \in \mathbb{Z}$.

Solution 3

When $n \geq 2$, we have $f(n) = f(n-1) + n + 3$. Rearranging, we obtain $f(n) - f(n-1) = n + 3$. Using $f(n) - f(n-1) = n + 3$ with integer values of n from 100 to 2, we have

$$\begin{aligned} f(100) - f(99) &= 100 + 3 \\ f(99) - f(98) &= 99 + 3 \\ f(98) - f(97) &= 98 + 3 \\ &\vdots \\ f(3) - f(2) &= 3 + 3 \\ f(2) - f(1) &= 2 + 3 \end{aligned}$$

If we add all of the terms on the left side of each equal sign and then simplify, we are left with only $f(100) - f(1)$. If we add all of the terms on the right side of each equal sign, we get

$$2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3)$$

Therefore,

$$\begin{aligned} f(100) - f(1) &= 2 + 3 + 4 + \cdots + 98 + 99 + 100 + 99(3) \\ &= (1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100) - 1 + 99(3) \\ &= \frac{100(101)}{2} - 1 + 99(3) \\ &= 5050 - 1 + 297 \\ &= 5346 \end{aligned}$$

Since $f(1) = 50$, we have $f(100) - 50 = 5346$, and so $f(100) = 5396$.

That is, 5396 cans are collected on the 100th day.

EXTENSION: Assuming their target is met each day of the 100-day campaign, how many cans of food will they collect in total?