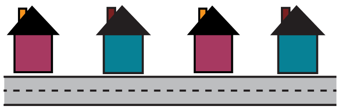

Problem of the Week
Problem E and Solution
A Street Called Square

Problem

Alyssa, Bilal, Constantine, and Daiyu each live in a house that is numbered with a positive integer.

A four-digit perfect square is formed by writing Alyssa's house number followed by Bilal's house number. Another four-digit perfect square is formed by writing Constantine's house number followed Daiyu's house number. Constantine's house number is 31 more than Alyssa's house number, and Daiyu's house number is 31 more than Bilal's house number. What is each person's house number?

Solution

Both Alyssa's house number and Bilal's house number must be two-digit integers. If Alyssa's house number is a one-digit integer, Bilal's house number would have to be a three-digit integer to create the four-digit perfect square. This means that Constantine's house number would be a two-digit integer and Daiyu's house number would be at least a three-digit integer. Writing Constantine's house number followed by Daiyu's house number would create a five-digit integer. A similar argument can be presented if Bilal's house number is a one-digit integer. Therefore, both Alyssa and Bilal have house numbers that are two-digit integers.

Let Alyssa's house number be x and Bilal's house number be y. Then $100x + y$ is the four-digit integer created by writing Alyssa's house number followed by Bilal's house number. It is given that $100x + y$ is a perfect square, so $100x + y = k^2$ for some positive integer k.

Therefore, Constantine's house number is $(x + 31)$ and Daiyu's house number is $(y + 31)$. The number created by writing Constantine's house number followed by Daiyu's house number is $100(x + 31) + (y + 31)$. This four-digit number is also a perfect square. So $100(x + 31) + (y + 31) = m^2$, for some positive integer m. Simplifying, we have

$$100x + 3100 + y + 31 = m^2$$
$$100x + y + 3131 = m^2$$

Substituting $100x + y = k^2$, we obtain $k^2 + 3131 = m^2$ or $3131 = m^2 - k^2$.

Since $m^2 - k^2$ is a difference of squares, we have $m^2 - k^2 = (m + k)(m - k) = 3131$.

Since m and k are positive integers, $m+k$ is a positive integer and $m+k>m-k$. Also, $m-k$ must be a positive integer since $(m+k)(m-k)=3131$. So we are looking for two positive integers that multiply to 3131. There are two possibilities, 3131×1 or 101×31.

First we will examine $(m+k)(m-k)=3131\times 1$. From this we obtain two equations in two unknowns, namely $m+k=3131$ and $m-k=1$. Subtracting the second equation from the first gives $2k=3130$ or $k=1565$. Then $k^2=1565^2=2\,449\,225$. This is not a four-digit number, so 3131×1 is not an admissible factorization of 3131.

Next we examine $(m+k)(m-k)=101\times 31$. This leads to $m+k=101$ and $m-k=31$. Subtracting the second equation from the first, we get $2k=70$ or $k=35$. Then $100x+y=k^2=1225$. Therefore, $x=12$ and $y=25$, since 1225 is the four-digit number formed by writing Alyssa's house number, x, followed by Bilal's house number, y.

Now, Constantine's house number is $x+31=12+31=43$ and Daiyu's house number is $y+31=25+31=56$. Notice that $4356=66^2$, so it is a four-digit perfect square.

Therefore, Alyssa's house number is 12, Bilal's house number is 25, Constantine's house number is 43, and Daiyu's house number is 56.