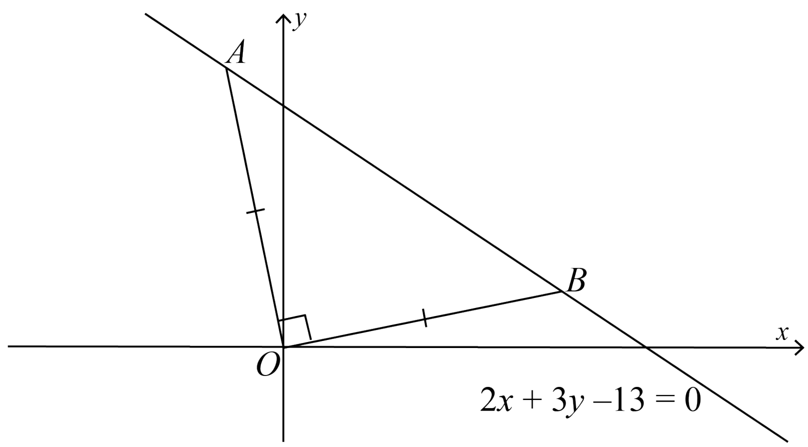
Problem of the Week
Problem E and Solution
Tilted Triangle

Problem

Suppose that $\triangle OAB$ is such that vertex O is located at the origin, and vertices A and B lie on the line $2x + 3y - 13 = 0$ with $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.



Solution

By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$.

Since points A and B lie on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$.

Since $OA = OB$, we have

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
13a^2 - 13b^2 - 52a + 52b &= 0 \\
a^2 - b^2 - 4a + 4b &= 0
\end{aligned}$$

By factoring, we then obtain $(a+b)(a-b) - 4(a-b) = 0$ or $(a-b)(a+b-4) = 0$.

This implies that, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$.

From here we proceed with two different solutions.

Solution 1

Since B has coordinates $\left(b, \frac{-2b+13}{3}\right)$, the slope of OB is equal to $\dfrac{\frac{-2b+13}{3}}{b} = \dfrac{-2b+13}{3b}$.

Since A has coordinates $\left(a, \frac{-2a+13}{3}\right)$, the slope of OA is equal to $\dfrac{\frac{-2a+13}{3}}{a} = \dfrac{-2a+13}{3a}$.

Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of OA is the negative reciprocal of the slope of OB. Therefore,

$$\begin{aligned}\frac{-2a+13}{3a} &= -\frac{1}{\frac{-2b+13}{3b}} \\ &= \frac{-3b}{-2b+13}\end{aligned}$$

Simplifying, we obtain

$$\begin{aligned}(-2a+13)(-2b+13) &= (-3b)(3a) \\ 4ab - 26a - 26b + 169 &= -9ab \\ 13ab - 26a - 26b + 169 &= 0 \\ ab - 2a - 2b + 13 &= 0\end{aligned}$$

Substituting $a = 4 - b$, we have

$$\begin{aligned}(4-b)b - 2(4-b) - 2b + 13 &= 0 \\ 4b - b^2 - 8 + 2b - 2b + 13 &= 0 \\ b^2 - 4b - 5 &= 0\end{aligned}$$

Factoring, this becomes $(b-5)(b+1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$.

In each case, the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

Since $\triangle AOB$ is a right-angled triangle, we can use OB as the base and OA as the height in the formula for the area of a triangle.

Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26} \times \sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Solution 2

Since $a = 4 - b$, we can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4 - b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4 - b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is right-angled at O, by the Pythagorean Theorem, $AB^2 = OA^2 + OB^2$. Since $OA = OB$, this can be written $AB^2 = 2OB^2$. Thus,

$$\begin{aligned}(b - (4 - b))^2 + \left(\frac{-2b + 13}{3} - \frac{2b + 5}{3}\right)^2 &= 2\left(b^2 + \left(\frac{-2b + 13}{3}\right)^2\right)\\ (2b - 4)^2 + \left(\frac{-4b + 8}{3}\right)^2 &= 2\left(b^2 + \frac{4b^2 - 52b + 169}{9}\right)\\ 4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9}\end{aligned}$$

Multiplying by 9 and simplifying, we obtain

$$\begin{aligned}36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338\\ 52b^2 - 208b + 208 &= 26b^2 - 104b + 338\\ 26b^2 - 104b - 130 &= 0\\ b^2 - 4b - 5 &= 0\end{aligned}$$

Factoring, this becomes $(b - 5)(b + 1) = 0$. It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$.

When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$.

In each case, the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.

Since $\triangle AOB$ is a right-angled triangle, we can use OB as the base and OA as the height in the formula for the area of a triangle.

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