Problem of the Week

Problem E and Solution

Pizza Pi Box

Problem

A box in the shape of a triangle with side lengths 13, 14, and 15 is being used to pack a pizza pi. The pizza is in the shape of a circle and just touches each of the three sides of the triangular box. What is the area of the circle?

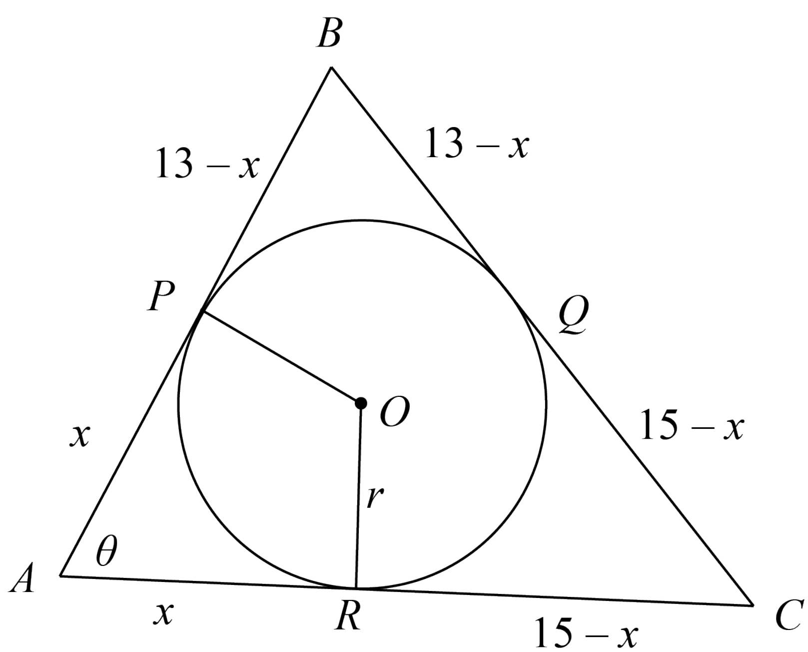
Solution

We label the triangle ABC, with $AB = 13$, $BC = 14$, and $AC = 15$.

Let P be the point where the circle touches the triangle on AB, Q be the point where the circle touches the triangle on BC, and R be the point where the circle touches the triangle on AC.

Let $\angle BAC = \theta$, and let the circle be centred at point O with radius r.

Using the second given result about circles, we know that $AP = AR$, $CR = CQ$, and $BP = BQ$. Let $AP = AR = x$. Then $BP = AB - AP = 13 - x$. Since $BP = BQ$, we have $BQ = BP = 13 - x$. Also, $CR = AC - AR = 15 - x$. Since $CR = CQ$, we have $CQ = 15 - x$.



Since $BC = 14$ and $BC = BQ + CQ$, we have $13 - x + 15 - x = 14$, thus $28 - 2x = 14$, and so $x = 7$.

Using the cosine law in $\triangle ABC$,

$$\begin{aligned}
BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\
14^2 &= 13^2 + 15^2 - 2(13)(15)\cos(\theta) \\
\cos\theta &= \frac{14^2 - 13^2 - 15^2}{-2(13)(15)} \\
&= \frac{-198}{-390} \\
&= \frac{33}{65}
\end{aligned}$$

Consider $\triangle AOR$ and $\triangle AOP$. Since $AR = AP$, $OR = OP = r$, and $AO = AO$ (same length), $\triangle AOR$ is congruent to $\triangle AOP$. Thus, $\angle OAP = \angle OAR = \frac{\theta}{2}$. Further, using the first given result about circles, we know that $\triangle AOR$ and $\triangle AOP$ are right-angled.

In $\triangle AOR$, $\tan\left(\frac{\theta}{2}\right) = \frac{r}{x} = \frac{r}{7}$. Thus, $r = 7\tan\left(\frac{\theta}{2}\right)$.

Using the identities $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, we obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2} = \frac{49}{65}$, and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} = \frac{16}{65}$.

Since $0 < \frac{\theta}{2} < 90^\circ$, $\cos\left(\frac{\theta}{2}\right) > 0$ and $\sin\left(\frac{\theta}{2}\right) > 0$. Thus, $\cos\left(\frac{\theta}{2}\right) = \frac{7}{\sqrt{65}}$ and $\sin\left(\frac{\theta}{2}\right) = \frac{4}{\sqrt{65}}$.

Therefore, we have

$$\begin{aligned}
r &= 7\tan\left(\frac{\theta}{2}\right) \\
&= \frac{7\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \\
&= \frac{7\left(\frac{4}{\sqrt{65}}\right)}{\frac{7}{\sqrt{65}}} \\
&= 4
\end{aligned}$$

Therefore, the area of the circle is $\pi(4)^2 = 16\pi$.