



Problem of the Week

Problem D and Solution

The Broken Screen PIN

Problem

Jordan dropped their phone, and now the touchscreen is badly damaged. The left and centre columns of the number pad are completely unresponsive. Only the right-hand column, containing the digits 3, 6, and 9, is still functioning. Jordan wants to set a new 4-digit lock screen PIN using only the working numbers. Jordan sets two rules for the PIN:

- A single digit can occur at most three times in the PIN (for example, 9999 is not allowed).
- The 4-digit PIN must be divisible by 9.

How many different 4-digit PINs can Jordan create?

Solution

Solution 1

In this solution we consider the possible sums that the digits of the PIN can add to.

The minimum digit sum would be created using three 3s and one 6, yielding a sum of 15. The maximum digit sum would be created using three 9s and one 6, yielding a sum of 33. Neither of these sums are divisible by 9. However, between 15 and 33, there are two numbers, namely 18 and 27, which are divisible by 9. We are therefore looking for four digits whose sum is either 18 or 27. The following table summarizes the possible digit combinations and the resulting sums.

Number of 3s	Number of 6s	Number of 9s	Digit Sum	Divisible by 9?
3	1	0	$3 + 3 + 3 + 6 = 15$	No
3	0	1	$3 + 3 + 3 + 9 = 18$	Yes
1	3	0	$3 + 6 + 6 + 6 = 21$	No
0	3	1	$6 + 6 + 6 + 9 = 27$	Yes
1	0	3	$3 + 9 + 9 + 9 = 30$	No
0	1	3	$6 + 9 + 9 + 9 = 33$	No
2	2	0	$3 + 3 + 6 + 6 = 18$	Yes
2	0	2	$3 + 3 + 9 + 9 = 24$	No
0	2	2	$6 + 6 + 9 + 9 = 30$	No
2	1	1	$3 + 3 + 6 + 9 = 21$	No
1	2	1	$3 + 6 + 6 + 9 = 24$	No
1	1	2	$3 + 6 + 9 + 9 = 27$	Yes

Now, we can create the numbers using the information in the rows of the table in which the sum of the digits is divisible by 9.

- First we can create all four-digit numbers having three 3s and one 9. The numbers are 3339, 3393, 3933, and 9333. There are 4 possible numbers.



- Then we can create all four-digit numbers having three 6s and one 9. The numbers are 6669, 6696, 6966, and 9666. There are 4 possible numbers.
- We continue by creating the four-digit numbers which contain two 3s and two 6s. The numbers are 3366, 3636, 3663, 6633, 6363, and 6336. There are 6 possible numbers.
- Finally, we create all numbers containing two 9s, one 3 and one 6. The numbers are 9936, 9963, 9396, 9693, 9369, 9639, 3996, 6993, 3969, 6939, 3699, and 6399. These numbers can be created systematically by placing the 9s in the six possible ways in a four-digit number. Then, for each of these ways of placing the 9s, the 3 and 6 can be filled in the remaining two spots in two ways. There are a total of 12 possible numbers.

Combining the results, there are $4 + 4 + 6 + 12 = 26$ possible PINs.

Solution 2

In this solution, we consider cases based on the number of 9s in the PIN.

- **Case 1:** The number contains three 9s.

In this case the digit sum is 27 plus the value of the fourth digit.

- If the fourth digit is a 3, the digit sum is 30. Since 30 is not divisible by 9, there is no possible PIN using these digits.
- If the fourth digit is a 6, the digit sum is 33. Since 33 is not divisible by 9, there is no possible PIN using these digits.

This case produces no possible PINs.

- **Case 2:** The number contains two 9s.

In this case the digit sum is 18 plus the sum of the two remaining digits.

- If the two remaining digits are both 3s, the digit sum is 24. Since 24 is not divisible by 9, there is no possible PIN using these digits.
- If the two remaining digits are both 6s, the digit sum is 30. Since 30 is not divisible by 9, there is no possible PIN using these digits.
- If one of the remaining digits is a 3 and the other digit is a 6, the digit sum is 27. Since 27 is divisible by 9, possible PINs exist using these digits. There are twelve possible PINs: 9936, 9963, 9396, 9693, 9369, 9639, 3996, 6993, 3969, 6939, 3699, and 6399. These numbers can be created systematically by placing the 9s in the six possible ways in a four-digit number. Then, for each of these ways of placing the 9s, the 3 and 6 can be filled in the remaining two spots in two ways.

This case produces 12 possible PINs.

- **Case 3:** The number contains one 9.

In this case the digit sum is 9 plus the sum of the three remaining digits.

- If each of the three remaining digits is a 3, the digit sum is 18. Since 18 is divisible by 9, possible PINs exist using these digits. There are four possible PINs: 3339, 3393, 3933, and 9333.



- If each of the three remaining digits is a 6, the digit sum is 27. Since 27 is divisible by 9, possible PINs exist using three digits. There are four possible PINs: 6669, 6696, 6966, and 9666.
- If one of the remaining digits is a 3 and each of the other two remaining digits is a 6, the digit sum is 24. Since 24 is not divisible by 9, there is no possible PIN using these digits.
- If one of the remaining digits is a 6 and each of the other two remaining digits is a 3, the digit sum is 21. Since 21 is not divisible by 9, there is no possible PIN using these digits.

This case produces 8 possible PINs.

• **Case 4:** The number contains no 9s.

In this case the digit sum is the sum of the four remaining digits.

- If the number contains three 3s and one 6, the digit sum is 15. Since 15 is not divisible by 9, there is no possible PIN using these digits.
- If the number contains three 6s and one 3, the digit sum is 21. Since 21 is not divisible by 9, there is no possible PIN using these digits.
- If two of the digits are 6s and two of the digits are 3s, the digit sum is 18. Since 18 is divisible by 9, possible PINs exist using these digits. There are 6 possible numbers: 3366, 3636, 3663, 6633, 6363, and 6336.

This case produces 6 possible PINs.

Combining the results from the four cases, there are $0 + 12 + 8 + 6 = 26$ possible PINs that Jordan can create.