



Problem of the Week

Problem D and Solution

The Call Cascade

Problem

The POTW School is testing its cybersecurity incident response plan. If the school's communication system is compromised, employees must relay instructions directly to colleagues. The communication director informs up to three employees, each of whom informs up to three others, continuing until everyone has received the instructions.

If the POTW School has 100 employees (including the communication director) and uses a phone call tree system where each employee phones 0, 1, 2, or 3 other employees, determine the maximum number of employees who only receive the message and do not need to make any phone calls in their system.

Solution

It is important to note that in order to minimize the number of callers, we need to maximize the number of calls made by those who do make calls.

Once the director makes the initial three phone calls, four people (the director and three others) have the information. There are $100 - 4 = 96$ people left to contact.

The next three people make three calls each, for a total of 9 calls. Now 13 people have the information and 87 people still need to be contacted.

The next 9 people make 3 calls each, for a total of 27 calls. Now 40 people have the information and 60 people still need to be contacted.

From here, we will present two approaches for figuring out how many people are needed to contact the remaining 60 people.

- **Approach 1:** If the next 27 people make 3 calls each for a total of 81 calls, this is $81 - 60 = 21$ calls too many. This means $21 \div 3 = 7$ of the 27 people do not need to make any calls. Thus, only $27 - 7 = 20$ more people need to make calls.
- **Approach 2:** In order to reach the final 60 people, only $60 \div 3 = 20$ more people need to make calls because each person phones 3 people.

The total number of people required to make calls is therefore $1 + 3 + 9 + 20 = 33$.

Therefore, $100 - 33 = 67$ is the maximum number of employees who do not need to make any phone calls in the phone tree system.