Problem of the Week
Problem D and Solution
Mean, Median, Mode, and Max

Problem

A list of six positive integers has all of the following properties:

- the only integer in the list that occurs more than once is 8,
- its median is 9, and
- its average (mean) is 10.

What is the largest possible integer that could appear in the list?

Solution

We write the list of numbers in increasing order as a, b, c, d, e, f.

Since the average of the six numbers is 10, then their sum is $6(10) = 60$.

We know that the number 8 occurs at least twice in this list. We also know the median of the six numbers is 9. Since $8 < 9$, then either $a = b = 8$ or $b = c = 8$. We will look at each case separately.

- Case 1: $a = b = 8$
  Then since the numbers are integers and the median is 9, it follows that either $c = d = 9$ or $c = 8$ and $d = 10$. Since we're told the only integer in the list that occurs more than once is 8, we must have $c = 8$ and $d = 10$. Then the list can be written as 8, 8, 8, 10, e, f. Since the sum of the numbers is 60, $8 + 8 + 8 + 10 + e + f = 60$, so $34 + e + f = 60$, or $e + f = 26$. Since we want f to be as large as possible, we must make e as small as possible. The smallest possible value for e is 11. Thus $f = 26 - 11 = 15$.

- Case 2: $b = c = 8$
  Then since the numbers are integers and the median is 9, it follows that $d = 10$. Then the list can be written as a, 8, 8, 10, e, f. Since the sum of the numbers is 60, $a + 8 + 8 + 10 + e + f = 60$, so $a + e + f + 26 = 60$, or $a + e + f = 34$. Since we want f to be as large as possible, we must make a and e as small as possible. The smallest possible value for a is 1, and the smallest possible value for e is 11. Thus $f = 34 - 1 - 11 = 22$.

Therefore, the largest possible integer that could appear in the list is 22.