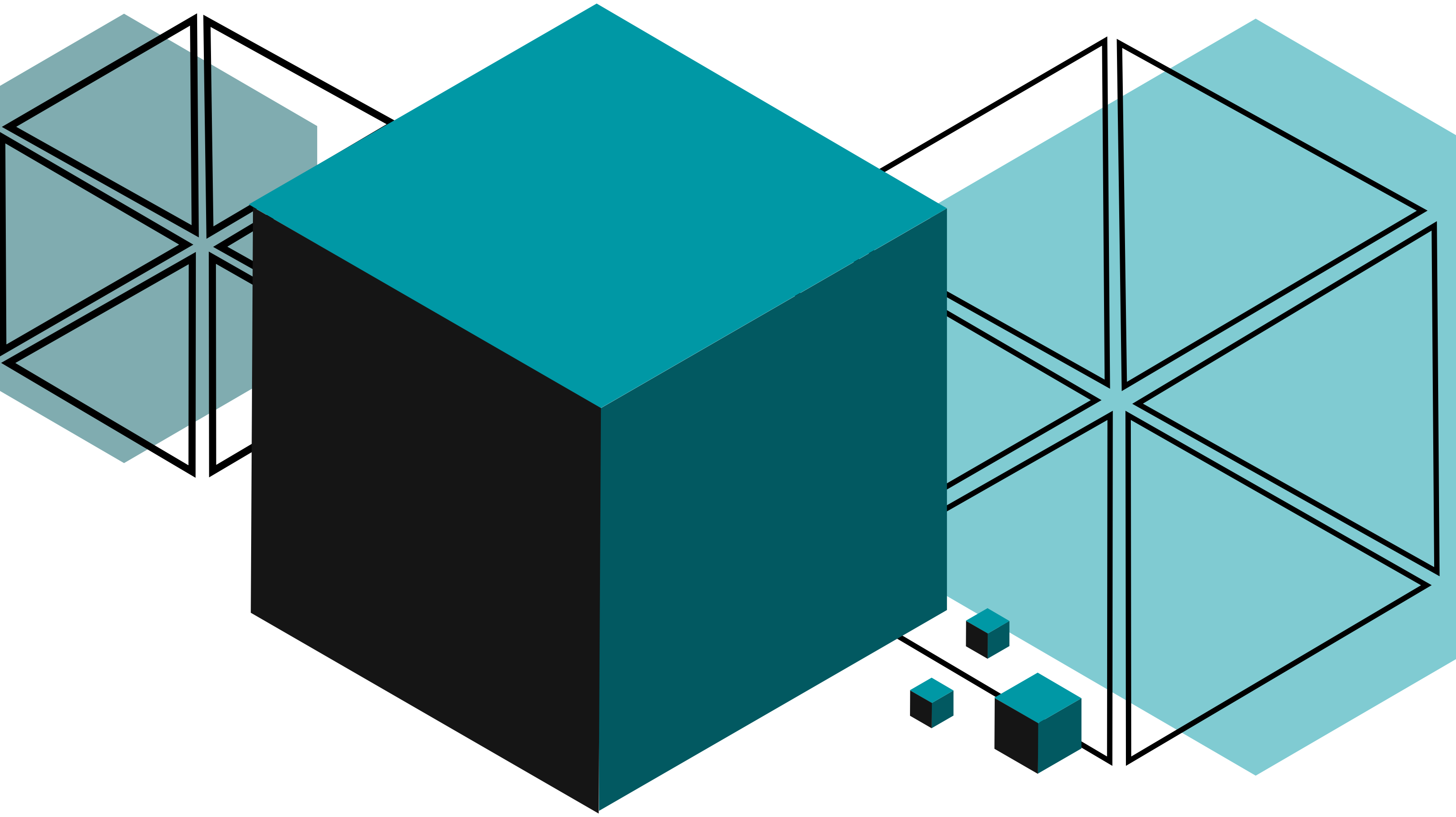
Problem of the Week

Problems and Solutions 2025-2026



Problem D

Grade 9/10

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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Problem of the Week

Problem D

The Twos and Fives Dilemma

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks of bills that each have a total value of exactly \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

Problem of the Week
Problem D and Solution
The Twos and Fives Dilemma

Problem

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks of bills that each have a total value of exactly \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

Solution

Consider a stack of bills with a total value of \$100 that includes x \$2 bills and y \$5 bills. The \$2 bills are worth $\$2x$ and the \$5 bills are worth $\$5y$, and so $2x + 5y = 100$.

Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$ or $5y = 100 - 2x$. (We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$2 bill and at least one \$5 bill.)

Since $x \geq 1$, then

$$\begin{aligned} 2x &\geq 2 \\ -2x &\leq -2 \\ 100 - 2x &\leq 100 - 2 \\ 100 - 2x &\leq 98 \end{aligned}$$

Also, since $5y = 100 - 2x$, this becomes $5y \leq 98$.

This means that $y \leq \frac{98}{5} = 19.6$. Since y is an integer, then $y \leq 19$.

Notice that since $5y = 100 - 2x$, then the right side is the difference between two even integers and is therefore even. This means that $5y$ (the left side) is even, which means that y must be even.

Since y is even, $y \geq 1$, and $y \leq 19$, then the possible values of y are 2, 4, 6, 8, 10, 12, 14, 16, and 18.

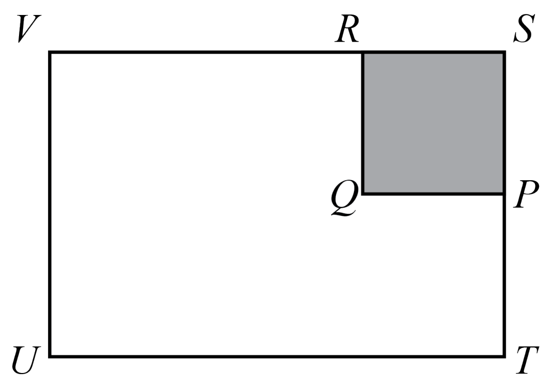
Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$. These ordered pairs are $(45, 2)$, $(40, 4)$, $(35, 6)$, $(30, 8)$, $(25, 10)$, $(20, 12)$, $(15, 14)$, $(10, 16)$, and $(5, 18)$.

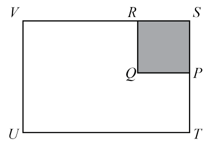
Therefore, the maximum number of stacks that Virat can create is 9.

Problem of the Week
Problem D
Cutting the Corner

Rectangle $STUV$ has P on ST, R on SV, and Q inside the rectangle such that $PQRS$ is a square. When square $PQRS$ is removed from rectangle $STUV$, the remaining shape has an area of 92 m^2.

If $PT = 4$ m and $RV = 8$ m, what is the area of rectangle $STUV$?





Problem of the Week

Problem D and Solution

Cutting the Corner

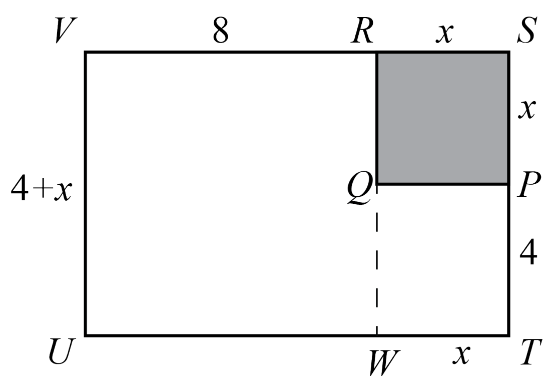
Problem

Rectangle $STUV$ has P on ST, R on SV, and Q inside the rectangle such that $PQRS$ is a square. When square $PQRS$ is removed from rectangle $STUV$, the remaining shape has an area of 92 m^2.

If $PT = 4$ m and $RV = 8$ m, what is the area of rectangle $STUV$?

Solution

Let x represent the side length of square $PQRS$. In the diagram, extend RQ to intersect TU at W. This creates rectangle $PTWQ$ and rectangle $RWUV$. Also, $UV = PT + SP = (4 + x)$ m and $TW = RS = x$ m.



We are given that area $PTWQ$ + area $RWUV = 92$ m^2. That is,

$$\begin{aligned} PT \times TW + RV \times UV &= 92 \\ 4x + 8(4 + x) &= 92 \\ 4x + 32 + 8x &= 92 \\ 12x + 32 &= 92 \\ 12x &= 60 \\ x &= 5 \end{aligned}$$

Thus, $x = 5$ m, and so $SV = 8 + x = 13$ m, and $UV = 4 + x = 9$ m.

Therefore, the area of rectangle $STUV$ is $SV \times UV = 13 \times 9 = 117$ m^2.

Problem of the Week

Problem D

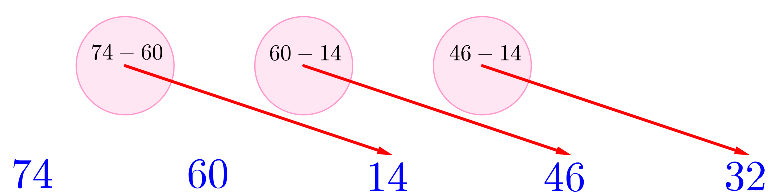
Difference Detective

The *non-negative difference* between two numbers a and b is $b - a$ or $a - b$, whichever is greater than or equal to 0. For example, the non-negative difference between 24 and 64 is 40.

Consider the sequence with first term 74, second term 60, and each term after equal to the non-negative difference between the previous two terms. That is, the third term is $74 - 60 = 14$, the fourth term is $60 - 14 = 46$, and the fifth term is $46 - 14 = 32$. So, the first five terms in the sequence are:

$$74, 60, 14, 46, 32$$

Determine the sum of the first 1306 terms in the sequence.



Problem of the Week

Problem D and Solution

Difference Detective

Problem

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Consider the sequence with first term 74, second term 60, and each term after equal to the non-negative difference between the previous two terms. That is, the third term is $74 - 60 = 14$, the fourth term is $60 - 14 = 46$, and the fifth term is $46 - 14 = 32$. So, the first five terms in the sequence are:

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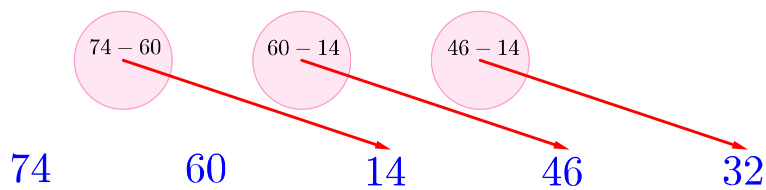


Solution

We start by generating more terms in the sequence in an attempt to find a pattern.

Using the rule for creating the sequence, we determine that the first 23 terms are

$$74,\ 60,\ 14,\ 46,\ 32,\ 14,\ 18,\ 4,\ 14,\ 10,\ 4,\ 6,\ 2,\ 4,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0$$

The first 14 terms in the sequence have no apparent pattern. Afterwards, the terms repeat 2, 2, 0. Since each term is determined by the previous two, this sequence of 2, 2, 0 will repeat indefinitely.

Thus, the first 14 terms in the sequence are followed by n groups of 2, 2, 0. What is the value of n? We want a total of 1306 terms. If we remove the first 14 terms, we require $1306 - 14 = 1292$ more terms. If we divide 1292 by 3 we are able to determine how many complete sequences of 2, 2, 0 are needed. Since $1292 \div 3 = 430\frac{2}{3}$, there will be 430 complete sequences of 2, 2, 0 followed by 2, 2.

The required sum is the sum of the first 14 terms plus $430 \times (2 + 2 + 0) + 2 + 2$. The sum of the first 14 terms is 302. Thus, the sum of the first 1306 terms in the sequence is $302 + 430 \times 4 + 4 = 2026$.

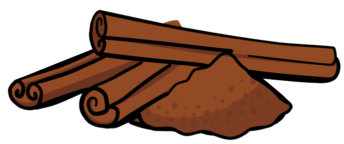
Therefore, the sum of the first 1306 terms in the sequence is 2026.

Problem of the Week

Problem D

A Sweet Mix

Mariana is buying cinnamon sugar, which is a mix made of cinnamon and sugar. There are two kinds available; one is in a red package and the other is in a blue package, but both packages have the same mass. In the red package, the ratio of cinnamon to sugar is $1 : 3$, by mass. In the blue package, the ratio of cinnamon to sugar is $1 : 4$, by mass. Mariana buys one red package and two blue packages, then mixes them together to create a new cinnamon sugar mix. Determine the ratio of cinnamon to sugar in Mariana's new cinnamon sugar mix.



Problem of the Week

Problem D and Solution

A Sweet Mix

Problem

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Solution

Solution 1

Let x represent the mass of each package of cinnamon sugar. Then the total mass of one red package of mix is x and the total mass of two blue packages of mix is $2x$.

Since the ratio of cinnamon to sugar in the red package, by mass, is $1:3$, then $\frac{1}{4}$ of the mass of the red package mix is cinnamon. It follows that the total mass of cinnamon in the red package mix is $\frac{1}{4}x$ and the total mass of sugar in the red package mix is $\frac{3}{4}x$.

Similarly, since the ratio of cinnamon to sugar in the blue package, by mass, is $1:4$, then $\frac{1}{5}$ of the mass of the blue package mix is cinnamon. It follows that the total mass of cinnamon in two blue packages of mix is $\frac{1}{5}(2x)=\frac{2}{5}x$ and the total mass of sugar in two blue packages of mix is $\frac{4}{5}(2x)=\frac{8}{5}x$.

When the packages are mixed to create the new cinnamon sugar mix, we obtain the following:

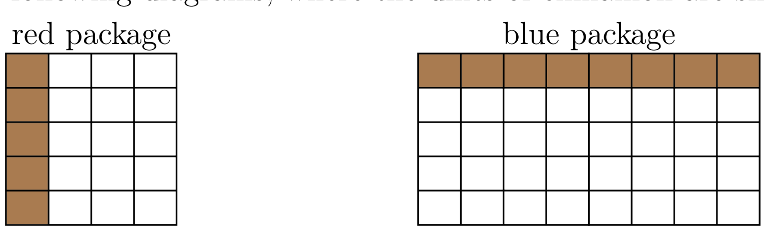
- Total mass of cinnamon: $\frac{1}{4}x+\frac{2}{5}x=\frac{5}{20}x+\frac{8}{20}x=\frac{13}{20}x$
- Total mass of sugar: $\frac{3}{4}x+\frac{8}{5}x=\frac{15}{20}x+\frac{32}{20}x=\frac{47}{20}x$

Thus, the ratio of cinnamon to sugar is $\frac{13}{20}x:\frac{47}{20}x=13:47$.

Solution 2

Let $\frac{1}{20}$ of each package be a unit of mass. Since the ratio of cinnamon to sugar in the red package, by mass, is $1:3$, then $\frac{1}{4}$ of the package, or 5 units of mass are cinnamon. Then the remaining 15 units are sugar.

Since there are two blue packages, then there are 40 units of mass in total of the blue package mix. Since the ratio of cinnamon to sugar in the blue package, by mass, is $1:4$, then $\frac{1}{5}$ of the mix, or 8 units of mass are cinnamon. Then the remaining 32 units are sugar. This is illustrated in the following diagrams, where the units of cinnamon are shaded.



When the packages are combined, there is a total of 60 units of mass. Of these, $5+8=13$ are cinnamon and $15+32=47$ are sugar. Thus, the ratio of cinnamon to sugar is $13:47$.

Problem of the Week

Problem D

Number Triangle

Consecutive positive integers are arranged in rows into the shape of a triangle. In this triangle, the top row contains the integer 1, and each row below the top row contains one more integer than the row above, starts with the next consecutive integer after the largest integer in the row above, and lists the consecutive integers in increasing order. That is, the first row contains the integer 1, the second row contains the integers 2 and 3, the third row contains the integers 4, 5, and 6, the fourth row contains the integers 7, 8, 9, and 10, and so on.

In which row will the integer 2026 appear? What integers are in this row?

$$\begin{array}{c} 1 \\ 2 \quad 3 \\ 4 \quad 5 \quad 6 \\ 7 \quad 8 \quad 9 \quad 10 \\ 11 \quad 12 \quad 13 \quad 14 \quad 15 \\ \vdots \end{array}$$

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Problem of the Week
Problem D and Solution
Number Triangle

Problem

Consecutive positive integers are arranged in rows into the shape of a triangle. In this triangle, the top row contains the integer 1, and each row below the top row contains one more integer than the row above, starts with the next consecutive integer after the largest integer in the row above, and lists the consecutive integers in increasing order. That is, the first row contains the integer 1, the second row contains the integers 2 and 3, the third row contains the integers 4, 5, and 6, the fourth row contains the integers 7, 8, 9, and 10, and so on.

In which row will the integer 2026 appear? What integers are in this row?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 3 & & & \\ & & 4 & & 5 & & 6 & & \\ & 7 & & 8 & & 9 & & 10 & \\ 11 & & 12 & & 13 & & 14 & & 15 \\ & & & & \vdots & & & & \end{array}$$

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution

After the first row, there is 1 integer in the triangle.

After the second row, there are $1 + 2 = 3$ integers in the triangle.

After the third row, there are $1 + 2 + 3 = 6$ integers in the triangle.

After the fourth row, there are $1 + 2 + 3 + 4 = 10$ integers in the triangle.

After the fifth row, there are $1 + 2 + 3 + 4 + 5 = 15$ integers in the triangle.

After the n^{th} row, there are $1 + 2 + 3 + \cdots + n$ integers in the triangle.

If 2026 is in row n, then $1 + 2 + 3 + \cdots + n \geq 2026$ and $1 + 2 + 3 + \cdots + (n - 1) < 2026$.

To determine n, let's try solving $1 + 2 + \cdots + n = 2026$.

It is given that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Thus, we are looking to solve $\frac{n(n+1)}{2} = 2026$, or equivalently $n^2 + n = 4052$.

At this point we will use trial and error. At the end of the solution, a more algebraic approach using the quadratic formula is presented.

Suppose $n = 25$. Then $n^2 + n = 25^2 + 25 = 650 < 4052$.

We can try a larger value for n. Suppose $n = 50$. Then, $n^2 + n = 50^2 + 50 = 2550 < 4052$.

We can try another yet larger value for n. Suppose $n = 65$. Then, $n^2 + n = 65^2 + 65 = 4290 > 4052$.

We are now close. We can try a smaller value for n. Suppose $n = 63$. Then, $n^2 + n = 63^2 + 63 = 4032 < 4052$.

Since this value is very close to 4052, we can then try the next integer, so let $n = 64$. Then, $n^2 + n = 64^2 + 64 = 4160 > 4052$.

This tells us that the integer 2026 is in the 64^{th} row. Further, the first 63 rows contain the integers from 1 to $\frac{63(64)}{2} = 2016$. Therefore, the 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.

We will finish by showing how we can find the value of n algebraically. We will first find the value of $n, n > 0$, so that

$$\begin{aligned} \frac{n(n+1)}{2} &= 2026 \\ n(n+1) &= 4052 \\ n^2 + n - 4052 &= 0 \end{aligned}$$

The quadratic formula can be used to solve for n.

$$\begin{aligned} n &= \frac{-1 \pm \sqrt{1 - 4(1)(-4052)}}{2} \\ n &= \frac{-1 \pm \sqrt{16\,209}}{2} \end{aligned}$$

It follows that $n \approx 63.157$ or $n \approx -64.157$. Since $n \approx -64.157 < 0$, it is inadmissible. Then $n \approx 63.157$.

Notice that $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2026$, and $1 + 2 + 3 + \cdots + 64 = \frac{64(65)}{2} = 2080 \geq 2026$.

Thus, the integer 2026 is in the 64^{th} row. The 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.

Problem of the Week

Problem D

The Flavour Foundry

Salt and Pepper are chefs scaling up a recipe for a large banquet. The total amount of dried garlic needed is 60% more than what Salt has in stock and 80% more than what Pepper has in stock. After combining the amounts they have in stock, they find they have more than necessary. What percentage over the required amount do they have? Round your answer to the nearest tenth of a percent.

Problem of the Week

Problem D and Solution

The Flavour Foundry

Problem

Salt and Pepper are chefs scaling up a recipe for a large banquet. The total amount of dried garlic needed is 60% more than what Salt has in stock and 80% more than what Pepper has in stock. After combining the amounts they have in stock, they find they have more than necessary. What percentage over the required amount do they have? Round your answer to the nearest tenth of a percent.

Solution

Let g be the total required amount needed in kg, s be Salt's amount, in kg, p be Pepper's amount, in kg.

Since the required amount is 60% more than the amount Salt has, then $g = 1.60s = \frac{8}{5}s$, or $s = \frac{5}{8}g$.

Similarly, since the required amount is 80% more than the amount Pepper has, $g = 1.80p = \frac{9}{5}p$, or $p = \frac{5}{9}g$.

Let t be the total amount Salt and Pepper have, in kg.

Now,

$$\begin{aligned} t &= s + p \\ &= \frac{5}{8}g + \frac{5}{9}g \\ &= \left(\frac{5}{8} + \frac{5}{9}\right) g \\ &= \left(\frac{45}{72} + \frac{40}{72}\right) g \\ &= \frac{85}{72}g \\ &= \left(1 + \frac{13}{72}\right) g \end{aligned}$$

Therefore, the total amount Salt and Pepper have exceeds the required amount by $\frac{13}{72} \times 100\%$, which is approximately 18.1%.

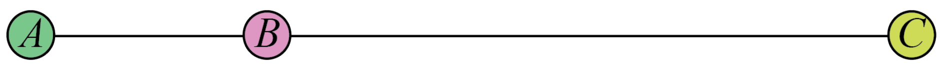
Problem of the Week

Problem D

Walking Away

Anna, Biljana, and Cees stand in a row in a large field. Anna is 180 m west of Biljana and Cees is 450 m east of Biljana. At the same time, Anna and Cees begin walking. Anna walks north at a constant rate of 75 m/min and Cees walks south at a constant rate of 90 m/min. Biljana does not move.

After how many minutes of walking will the distance between Biljana and Cees be twice the distance between Anna and Biljana?



Problem of the Week
Problem D and Solution
Walking Away

Problem

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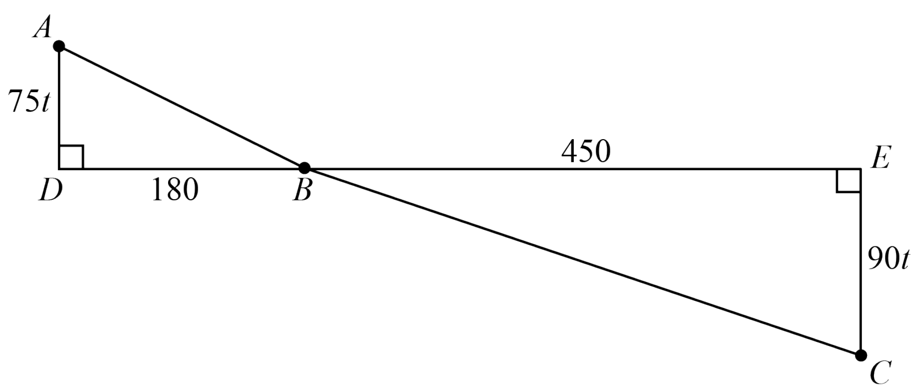
After how many minutes of walking will the distance between Biljana and Cees be twice the distance between Anna and Biljana?

Solution

Solution 1

Let t represent the number of minutes until the distance between Biljana and Cees is twice the distance between Anna and Biljana. In t minutes Anna will walk $75t$ m north and Cees will walk $90t$ m south. The following diagram shows Anna's position, A, Biljana's position, B, and Cees's position, C, in metres, at time $t > 0$.



Since both triangles in the diagram are right-angled triangles, we can use the Pythagorean Theorem to determine the value of t when $BC = 2AB$.

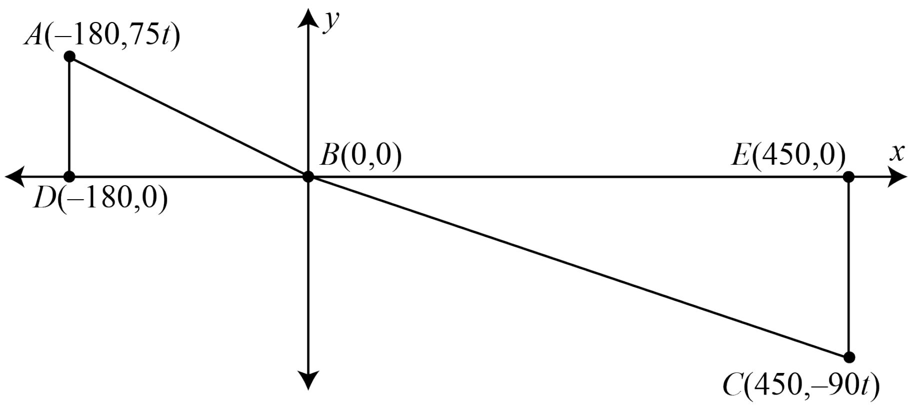
$$\begin{aligned}
BC &= 2AB \\
(BC)^2 &= (2AB)^2 \\
BE^2 + EC^2 &= 4(AD^2 + DB^2) \\
450^2 + (90t)^2 &= 4((75t)^2 + 180^2) \\
202\,500 + 8100t^2 &= 4(5625t^2 + 32\,400) \\
202\,500 + 8100t^2 &= 22\,500t^2 + 129\,600 \\
72\,900 &= 14\,400t^2 \\
\frac{81}{16} &= t^2
\end{aligned}$$

Since $t > 0$, it follows that $t = \frac{9}{4} = 2\frac{1}{4}$ min. Therefore, after $2\frac{1}{4}$ minutes of walking, the distance between Biljana and Cees will be twice the distance between Anna and Biljana.

Solution 2

In this solution, we represent Anna, Biljana and Cees's original positions as points on the x-axis so that Biljana is positioned at the origin $B(0,0)$, Anna is positioned 180 units left of Biljana at $D(-180,0)$ and Cees is positioned 450 units right of Biljana at $E(450,0)$.

Let t represent the number of minutes until the distance between Biljana and Cees is twice the distance between Anna and Biljana. In t minutes Anna will walk $75t$ m north to the point $A(-180,75t)$, and Cees will walk $90t$ m south to the point $C(450,-90t)$.



The distance from a point $P(x,y)$ to the origin can be found using the formula $d=\sqrt{x^2+y^2}$.

Then $AB=\sqrt{(-180)^2+(75t)^2}=\sqrt{32\,400+5625t^2}$ and
$BC=\sqrt{(450)^2+(-90t)^2}=\sqrt{202\,500+8100t^2}$.

$$\begin{aligned}
BC &= 2AB\\
\sqrt{202\,500+8100t^2} &= 2\sqrt{32\,400+5625t^2}\\
(\sqrt{202\,500+8100t^2})^2 &= (2\sqrt{32\,400+5625t^2})^2\\
202\,500+8100t^2 &= 4(32\,400+5625t^2)\\
202\,500+8100t^2 &= 129\,600+22\,500t^2\\
72\,900 &= 14\,400t^2\\
\frac{81}{16} &= t^2
\end{aligned}$$

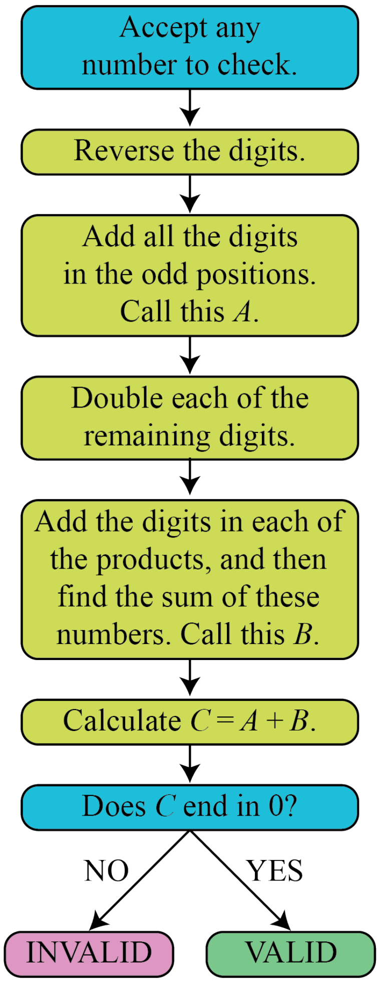
Since $t>0$, it follows that $t=\frac{9}{4}=2\frac{1}{4}$ min. Therefore, after $2\frac{1}{4}$ minutes of walking, the distance between Biljana and Cees will be twice the distance between Anna and Biljana.

Problem of the Week

Problem D

Check Please 2

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$

- Calculate B:

$$\begin{aligned} B &= (1 + 8) + (1 + 0) + 2 \\ &= 9 + 1 + 2 \\ &= 12 \end{aligned}$$

- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 8764 $X6X5$ $X6X8$ 5409 is a valid card number when verified by the Luhn Algorithm, where X is a single digit. Determine all possible values of X.

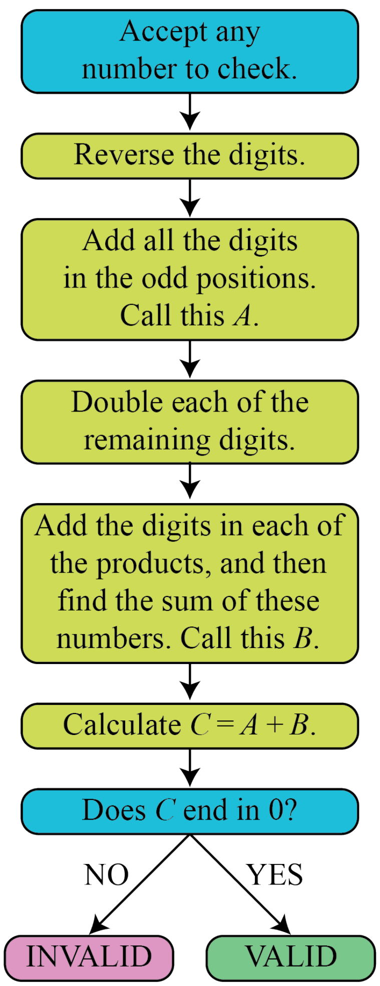
Problem of the Week

Problem D and Solution

Check Please 2

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



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- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$\begin{aligned} 2 \times 8 &= 16 \\ 2 \times 7 &= 14 \\ 2 \times 3 &= 6 \end{aligned}$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 8764 $X6X5$ $X6X8$ 5409 is a valid card number when verified by the Luhn Algorithm, where X is a single digit. Determine all possible values of X.

Solution

Solution 1

When the digits of the number are reversed the resulting number is $9045\ 8X6X\ 5X6X\ 4678$. The sum of the digits in the odd positions is

$$A = 9 + 4 + 8 + 6 + 5 + 6 + 4 + 7 = 49.$$

When the digits in the remaining positions are doubled, the following products are obtained:

$$2(0) = 0;\ 2(5) = 10;\ 2(X) = 2X;\ 2(X) = 2X;$$

$$2(X) = 2X;\ 2(X) = 2X;\ 2(6) = 12;\ 2(8) = 16$$

Let n represent the sum of the digits of $2X$.

When the digit sums from each of the products are added, the sum is:

$$B = 0 + (1 + 0) + n + n + n + n + (1 + 2) + (1 + 6) = 0 + 1 + 4n + 3 + 7 = 4n + 11.$$

Since $C = A + B$, we have $C = 49 + 4n + 11 = 60 + 4n$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what possible sums can be obtained? We summarize these in the following table.

Original Digit (X)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2X$)	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2X$ (n)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of $2X$ can be any integer from 0 to 9 inclusive. It follows that the only values for n are the integers from 0 to 9.

To be a valid number, the units digit of C must be 0. We want $60 + 4n$ to be an integer whose units digit is 0. Since the minimum value of n is 0, it follows that $60 + 4n \geq 60$. We will now look at cases based on the value of $60 + 4n$.

- Case 1: $60 + 4n = 60$.
  Then $4n = 0$, so $n = 0$. From our table, this occurs when $X = 0$. This is one possible value for X.

- Case 2: $60 + 4n = 70$.
  Then $4n = 10$, so $n = 2.5$. However n must be an integer value so this is not possible.

- Case 3: $60 + 4n = 80$.
  Then $4n = 20$, so $n = 5$. From our table, this occurs when $X = 7$. This is one possible value for X.
- Case 4: $60 + 4n = 90$.
  Then $4n = 30$, so $n = 7.5$. However n must be an integer value so this is not possible.
- Case 5: $60 + 4n = 100$.
  Then $4n = 40$, so $n = 10$. However n must be an integer from 0 to 9 inclusive, so this is not possible. This tells us that $60 + 4n < 100$, so there are no further cases.

Therefore, the two valid possibilities for X are 0 and 7.

When $X = 0$, the number is 8764 0605 0608 5409, which is indeed valid by the Luhn Algorithm.
When $X = 7$, the number is 8764 7675 7678 5409 which is indeed valid by the Luhn Algorithm.

Solution 2

The second solution looks at each of the possible values of X and then verifies the resulting number. A computer program or spreadsheet could be developed to solve this problem efficiently.

Remember that A is the sum of the digits in the odd positions of the reversal. Each of the digits in the even positions of the reversal are doubled, and B is the sum of the sum of the digits of each of these products. C is the sum $A + B$.

X	Number	Reversal	A	Double Even Digits	B	C	Valid?
0	8764 0605 0608 5409	9045 8060 5060 4678	49	0, 10, 0, 0, 0, 0, 12, 16	11	60	Yes
1	8764 1615 1618 5409	9045 8161 5161 4678	49	0, 10, 2, 2, 2, 2, 12, 16	19	68	No
2	8764 2625 2628 5409	9045 8262 5262 4678	49	0, 10, 4, 4, 4, 4, 12, 16	27	76	No
3	8764 3635 3638 5409	9045 8363 5363 4678	49	0, 10, 6, 6, 6, 6, 12, 16	35	84	No
4	8764 4645 4648 5409	9045 8464 5464 4678	49	0, 10, 8, 8, 8, 8, 12, 16	43	92	No
5	8764 5655 5658 5409	9045 8565 5565 4678	49	0, 10, 10, 10, 10, 10, 12, 16	15	64	No
6	8764 6665 6668 5409	9045 8666 5666 4678	49	0, 10, 12, 12, 12, 12, 12, 16	23	72	No
7	8764 7675 7678 5409	9045 8767 5767 4678	49	0, 10, 14, 14, 14, 14, 12, 16	31	80	Yes
8	8764 8685 8688 5409	9045 8868 5868 4678	49	0, 10, 16, 16, 16, 16, 12, 16	39	88	No
9	8764 9695 9698 5409	9045 8969 5969 4678	49	0, 10, 18, 18, 18, 18, 12, 16	47	96	No

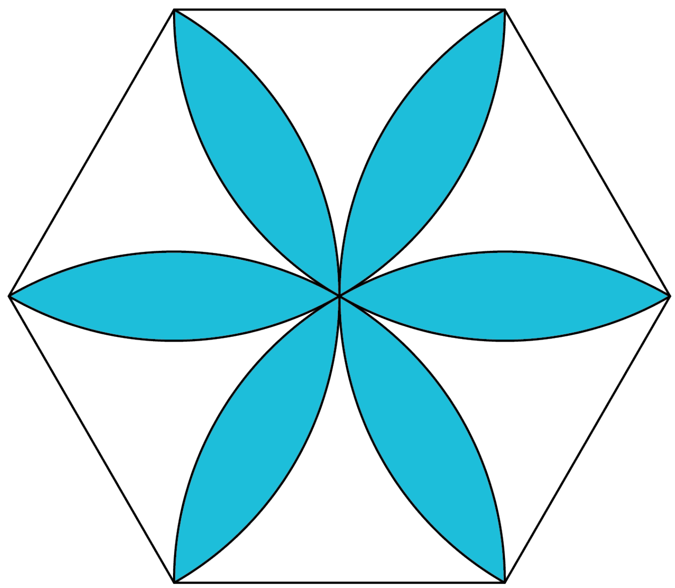
Therefore, the two valid possibilities for X are 0 and 7.

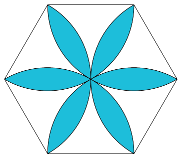
Problem of the Week

Problem D

A New Pi Plate

A pi-plate is in the shape of a regular hexagon with side length 8, and has an interesting pattern on it. From each vertex in the hexagon, a circular arc with centre at the vertex and radius 8 is drawn. This results in a flower-like pattern inside the hexagon, which is then shaded. Determine the total area of the shaded region.





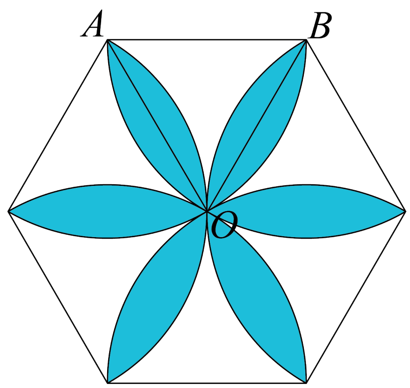
Problem of the Week
Problem D and Solution
A New Pi Plate

Problem

A pi-plate is in the shape of a regular hexagon with side length 8, and has an interesting pattern on it. From each vertex in the hexagon, a circular arc with centre at the vertex and radius 8 is drawn. This results in a flower-like pattern inside the hexagon, which is then shaded. Determine the total area of the shaded region.

Solution

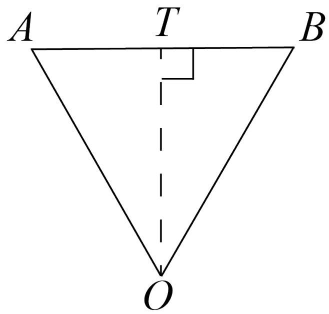
We label the centre of the hexagon O and consider two adjacent vertices, A and B. We draw in OA and OB.



We note that a regular hexagon is composed of six congruent equilateral triangles (the proof of this is left up to the student). Thus, $\triangle AOB$ is an equilateral triangle with side length 8.

To determine the total shaded area, we can consider the area of one-half of each 'petal' of the flower, and then multiply this area by 12, since there are 12 half-petals in the diagram. To determine the area of the region between the arc through B and O and the line segment OB, we can calculate the area of the sector AOB, with centre A and radius AB, and subtract the area of $\triangle AOB$.

We first determine the area of $\triangle AOB$. Construct altitude OT. Since $\triangle AOB$ is equilateral, it follows that OT bisects AB.



Since $AB = 8$, it follows that $AT = 4$. By the Pythagorean Theorem in $\triangle ATO$, $AO^2 = AT^2 + OT^2$. Therefore, $8^2 = 4^2 + OT^2$, and $OT^2 = 64 - 16 = 48$ follows. Since $OT > 0$, we have $OT = \sqrt{48}$.

Therefore, the area of $\triangle AOB$ is $\dfrac{(AB) \times (OT)}{2} = \dfrac{8 \times \sqrt{48}}{2} = 4\sqrt{48} = 16\sqrt{3}$.

We now determine the area of the sector AOB. Since $\angle OAB = 60^\circ$, the area of sector AOB is $\frac{1}{6}$ of the area of a circle with radius 8. Thus, the area of the sector is equal to $\frac{1}{6}\pi(AB)^2 = \frac{1}{6}\pi(8)^2 = \frac{32}{3}\pi$.

Therefore, the area of the shaded region between the arc through B and O and the line segment OB is $\left(\frac{32}{3}\pi - 16\sqrt{3}\right)$ units^2.

The original pi-plate has 12 of these regions, so the entire shaded area is $12 \times (\frac{32}{3}\pi - 16\sqrt{3}) = 128\pi - 192\sqrt{3} \approx 69.57$ units^2.

Problem of the Week

Problem D

Sharing Beads

Golriz and Diane each have some beads. If Diane gives six of her beads to Golriz, then Golriz would then have twice as many beads as Diane. Instead, if Diane takes six beads from Golriz, then both would have the same number of the beads. What is the total number of beads that Golriz and Diane have?

Problem of the Week

Problem D and Solution

Sharing Beads

Problem

Golriz and Diane each have some beads. If Diane gives six of her beads to Golriz, then Golriz would then have twice as many beads as Diane. Instead, if Diane takes six beads from Golriz, then both would have the same number of the beads. What is the total number of beads that Golriz and Diane have?

Solution

Solution 1

Let d be the number of beads that Diane has, and let g be the number of beads that Golriz has.

If Diane gives 6 beads to Golriz, then Diane would have $d-6$ beads and Golriz would have $g+6$ beads. Then Golriz has twice as many beads as Diane, meaning $g+6=2(d-6)$. Thus, $g=2d-18$.

If Diane takes 6 beads from Golriz, then Diane would have $d+6$ beads and Golriz would have $g-6$ beads. Then they both would have the same number of beads, meaning $d+6=g-6$. Thus, $g=d+12$.

Since $g=2d-18$ and $g=d+12$, we have $2d-18=d+12$ or $d=30$. Thus, $g=d+12=30+12=42$.

Thus, the total number of beads that Golriz and Diane have is $30+42=72$.

Solution 2

Let d be the number of beads that Diane has.

If Diane takes 6 beads from Golriz, then the two of them would have the same number of beads. This tells us that Golriz has 12 more beads than Diane, or that Golriz has $d+12$ beads.

If Diane gives 6 beads to Golriz, then Diane would have $d-6$ beads and Golriz would have $d+12+6=d+18$ beads.

From the given information, $d+18=2(d-6)$. Solving, we have $d+18=2d-12$ or $d=30$.

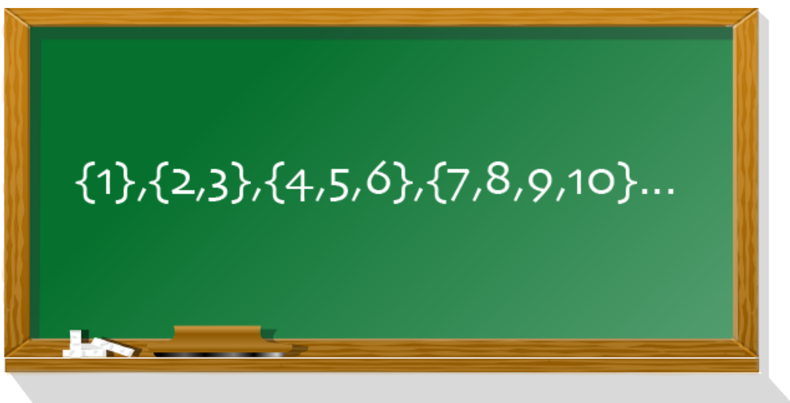
Thus, Diane has 30 beads and Golriz has $30+12=42$ beads. Therefore, they have $30+42=72$ beads in total.

Problem of the Week
Problem D
Counting Consecutively

Consider the sets of integers $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10\}$, $\ldots$.

These sets contain consecutive integers. The first set contains the integer 1, and each set thereafter contains one more integer than the previous set, with its smallest integer being one greater than the largest integer in the previous set.

Determine the sum of the integers in the 101^{st} set.

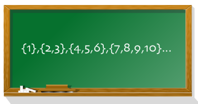


Note:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For example, $1 + 2 + 3 + 4 + 5 = 15$, and $\dfrac{5(6)}{2} = 15$.

Also, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, and $\dfrac{8(9)}{2} = 36$.



Problem of the Week

Problem D and Solution

Counting Consecutively

Problem

Consider the sets of integers $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, $\{7,8,9,10\}$, $\ldots$.

These sets contain consecutive integers. The first set contains the integer 1, and each set thereafter contains one more integer than the previous set, with its smallest integer being one greater than the largest integer in the previous set.

Determine the sum of the integers in the 101^{st} set.

NOTE:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

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For example, $1+2+3+4+5=15$, and $\dfrac{5(6)}{2}=15$.

Also, $1+2+3+4+5+6+7+8=36$, and $\dfrac{8(9)}{2}=36$.

Solution

Solution 1

Consider the sequence of the largest numbers in each set: 1, 3, 6, 10, $\ldots$.

This sequence of numbers consists of *triangular* numbers, which represent the sum of the first n natural numbers. That is, $1=1$, $3=1+2$, $6=1+2+3$, $10=1+2+3+4$. In general, the n^{th} triangular number, T_n, is the sum of the first n integers from 1 to n. That is, $T_n=1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Since each set contains one more integer than the previous set, the largest number in the n^{th} set will be equal to T_n. That is, the largest number in the n^{th} set will be equal to $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Therefore, the largest integer in the 100^{th} set is $T_{100}=\frac{100(100+1)}{2}=5050$. Thus, the smallest integer in the 101^{st} set is $5050+1=5051$.

In the 101^{st} set there are 101 integers and the smallest integer is 5051. Therefore, the largest integer is $5050+101=5151$.

Therefore, sum of the elements in the 101^{st} set is equal to

$$\begin{aligned}5051+5052+\cdots+5151&=(1+2+\cdots+5151)-(1+2+\cdots 5050)\\&=\frac{5151(5152)}{2}-\frac{5050(5051)}{2}\\&=13\,268\,976-12\,753\,775\\&=515\,201\end{aligned}$$

Solution 2

This solution uses quadratics, which may be beyond what some students have learned.

Consider then the sequence of largest numbers in each set: 1, 3, 6, 10, ...

The first differences of this sequence are 2, 3, 4, ... If we calculate the second differences, each is 1. We will leave it to the reader to justify that the second difference is a constant of 1 for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $f(x)$ is the largest number in the x^{th} set.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 1$, so

$$a + b + c = 1 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 3$, so

$$4a + 2b + c = 3 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 6$, so

$$9a + 3b + c = 6 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 2$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 3$, which we call equation (5).

We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 2$, and so $b = 2 - \frac{3}{2} = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{1}{2}$ into equation (1), we have $\frac{1}{2} + \frac{1}{2} + c = 1$, and so $c = 0$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$.

Therefore, the largest number in the 100^{th} set is $f(100) = \frac{1}{2}(100)^2 + \frac{1}{2}(100) = 5000 + 50 = 5050$.

The largest number in the 101^{st} set is $f(101) = \frac{1}{2}(101)^2 + \frac{1}{2}(101) = 5100.5 + 50.5 = 5151$.

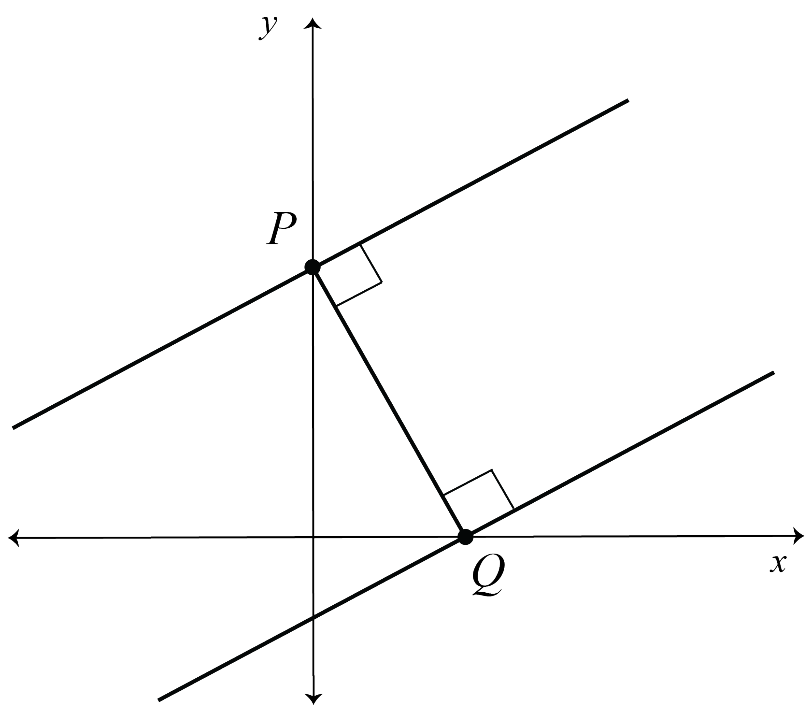
Therefore, sum of the elements in the 101^{st} set is equal to

$$\begin{aligned}
5051 + 5052 + \cdots + 5151 &= (1 + 2 + \cdots + 5151) - (1 + 2 + \cdots 5050) \\
&= \frac{5151(5152)}{2} - \frac{5050(5051)}{2} \\
&= 13\,268\,976 - 12\,753\,775 \\
&= 515\,201
\end{aligned}$$

Problem of the Week
Problem D
Parallel Puzzle

Two distinct lines are drawn such that the first line passes through point P on the positive y-axis and the second line passes through point Q on the positive x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



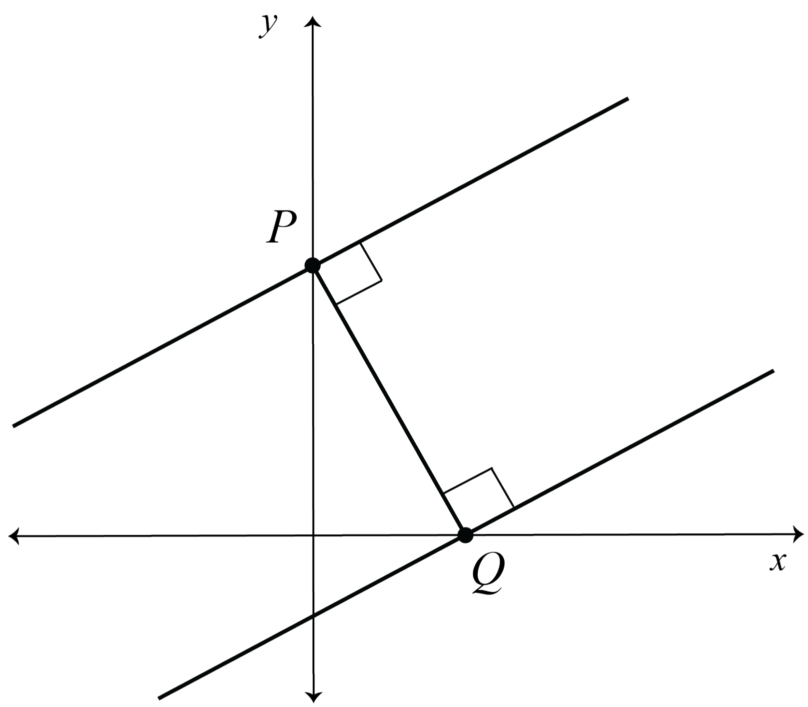
SUGGESTION: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

Problem of the Week
Problem D and Solution
Parallel Puzzle

Problem

Two distinct lines are drawn such that the first line passes through point P on the positive y-axis and the second line passes through point Q on the positive x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



SUGGESTION: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

Solution

We let l_1 represent the first line, which passes through P, and let l_2 represent the second line, which passes through Q.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P has coordinates $(0, k)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{m}$. Since k is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{m}x + k$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of the line through PQ is mk and Q has coordinates $(mk, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is m. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = mx + b$. Since $Q(mk, 0)$ lies on l_2, we have $0 = m(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2. A solution follows.

Let l_1 represent the line with equation $y = 4x + 3$. Let l_2 represent the second line, which passes through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P has coordinates $(0, 3)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{4}$. Since 3 is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{4}x + 3$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result $x = 12$ follows. Therefore, the x-intercept of the line through PQ is 12 and Q has coordinates $(12, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is 4. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = 4x + b$. Since $Q(12, 0)$ lies on l_2, we have $0 = 4(12) + b$ which simplifies to $b = -48$.

Therefore, the equation of l_2 is $y = 4x - 48$ and the y-intercept of l_2 is -48.

Problem of the Week

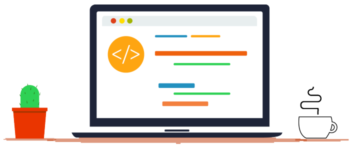
Problem D

One Thousand Inputs

Yeseo has written a program. After she inputs a number into the program the number is multiplied by 3, then 4 is added to the product, then the sum is divided by 3. The final result is then printed.

Yeseo inputs the number 2 into the program. Following the steps outlined, the number 2 is first multiplied by 3 to obtain 6. Then 4 is added to the product to obtain 10. Finally, this sum is divided by 3 to obtain $\frac{10}{3}$. The number $\frac{10}{3}$ is then printed.

Yeseo then inputs the output $\frac{10}{3}$ into the program. If Yeseo continues this process of taking the output number and using it as the next input, then what will be the 1000^{th} number she inputs into the program?



Problem of the Week
Problem D and Solution
One Thousand Inputs

Problem

Yeseo has written a program. After she inputs a number into the program the number is multiplied by 3, then 4 is added to the product, then the sum is divided by 3. The final result is then printed.

Yeseo inputs the number 2 into the program. Following the steps outlined, the number 2 is first multiplied by 3 to obtain 6. Then 4 is added to the product to obtain 10. Finally, this sum is divided by 3 to obtain $\frac{10}{3}$. The number $\frac{10}{3}$ is then printed.

Yeseo then inputs the output $\frac{10}{3}$ into the program. If Yeseo continues this process of taking the output number and using it as the next input, then what will be the 1000^{th} number she inputs into the program?

Solution

One way to solve this problem would be to work through all the different input numbers that Yeseo would have by following the steps in the program. However, since we are looking for the 1000^{th} input number, this is definitely not the most efficient solution.

Instead, we will first determine the first few input numbers. These are shown in the table.

Input	After Multiplying by 3	After Adding 4	After Dividing by 3
2	6	10	$\frac{10}{3}$
$\frac{10}{3}$	10	14	$\frac{14}{3}$
$\frac{14}{3}$	14	18	$\frac{18}{3}=6$
6	18	22	$\frac{22}{3}$
$\frac{22}{3}$	22	26	$\frac{26}{3}$
$\frac{26}{3}$	26	30	$\frac{30}{3}=10$

So the first seven input numbers are 2, $\frac{10}{3}$, $\frac{14}{3}$, 6, $\frac{22}{3}$, $\frac{26}{3}$, and 10. It appears as though the input numbers are increasing by $\frac{4}{3}$ each time. We can show this algebraically. Suppose an input number is p. Then following the steps in the program, we first multiply by 3 to obtain $3p$. We then add 4 to obtain $3p+4$. Finally, we divide by 3 to obtain $\frac{3p+4}{3}$. This can be simplified as follows:

$$\frac{3p+4}{3}=\frac{3p}{3}+\frac{4}{3}=p+\frac{4}{3}$$

Thus, for any input number p, the output (and next input number) will be $p+\frac{4}{3}$.

We will let t_1 be the first input number, t_2 be the second input number, and so on. Thus, we want to find t_{1000}. Using this notation, we now show three different solutions.

Solution 1
Since the difference between consecutive input numbers is $\frac{4}{3}$, then to get from t_1 to t_{1000}, we must add $\frac{4}{3}$ a total of 999 times. Therefore,

$$\begin{aligned} t_{1000} &= t_1 + 999 \times \frac{4}{3} \\ &= 2 + 999 \times \frac{4}{3} \\ &= 2 + 1332 \\ &= 1334 \end{aligned}$$

Thus, the 1000^{th} input number is 1334.

Solution 2
Since the difference between consecutive input numbers is $\frac{4}{3}$, it follows that the difference between every third input number is $3 \times \frac{4}{3} = 4$. Since the first input number, 2, is an integer, it follows that every third input number after that will also be an integer, and that these integer values form a sequence that is increasing by 4 each time. Since $1000 = 1 + 3 \times 333$, then starting from t_1, we must add 4 a total of 333 times to reach t_{1000}. Therefore,

$$\begin{aligned} t_{1000} &= t_1 + 333 \times 4 \\ &= 2 + 333 \times 4 \\ &= 1334 \end{aligned}$$

Thus, the 1000^{th} input number is 1334.

Solution 3
Since the difference between consecutive input numbers is $\frac{4}{3}$, if all input numbers are written with a denominator of 3, then the difference between the numerators of consecutive input numbers is 4. Thus, we can think of the numerators of consecutive input numbers as a sequence that starts at 6 (since $2 = \frac{6}{3}$) and increases by 4 each time. This sequence can be written as $4n + 2$, where $n = 1$ represents the first input number, $n = 2$ represents the second input number, and so on. Thus, $t_n = \frac{4n+2}{3}$ for any $n \geq 1$. Therefore,

$$t_{1000} = \frac{4(1000) + 2}{3} = \frac{4002}{3} = 1334$$

Thus, the 1000^{th} input number is 1334.

Computational Thinking (C)

Take me to the cover

Problem of the Week

Problem D

Sports Day

There were 500 students at a sports day camp. For the morning session, each student could choose between soccer or basketball. For the afternoon session, each student could choose between hockey or frisbee. Each student had to choose exactly one sport per session.

The following information is known about the sessions:

- 32% of the students chose basketball,
- $\frac{1}{5}$ of the students who chose hockey also chose basketball, and
- of the students who chose basketball, 50 more chose frisbee than hockey.

Determine the percentage of students who choose both soccer and frisbee.

Problem of the Week

Problem D and Solution

Sports Day

Problem

There were 500 students at a sports day camp. For the morning session, each student could choose between soccer or basketball. For the afternoon session, each student could choose between hockey or frisbee. Each student had to choose exactly one sport per session.

The following information is known about the sessions:

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- of the students who chose basketball, 50 more chose frisbee than hockey.

Determine the percentage of students who choose both soccer and frisbee.

Solution

There were 500 students in total and 32% of them chose basketball. Thus, $0.32 \times 500 = 160$ students chose basketball. Since students had to choose either basketball or soccer, it follows that the remaining $500 - 160 = 340$ students chose soccer. The table shows the information we have determined so far about the number of students who chose each sport.

	Soccer	**Basketball**	**Total**
Hockey			
Frisbee			
Total	340	160	500

We know that of the 160 students who chose basketball, 50 more chose frisbee than hockey. So, if h students chose basketball and hockey, then $h + 50$ students chose basketball and frisbee. Since $h + (h + 50) = 160$, it follows that $2h + 50 = 160$, so $2h = 110$, which gives $h = 55$. Thus, 55 of the students who chose basketball also chose hockey. Then the remaining $160 - 55 = 105$ students who chose basketball chose frisbee. We can now update our table accordingly.

	Soccer	**Basketball**	**Total**
Hockey		55	
Frisbee		105	
Total	340	160	500

Next, we know that $\frac{1}{5}$ of the students who chose hockey also chose basketball. Since 55 students chose hockey and basketball, this tells us that 55 is $\frac{1}{5}$ of the number of students who chose hockey. Thus, the number of students who chose hockey is $5 \times 55 = 275$. From here we can determine that the number of students who chose soccer and hockey is $275 - 55 = 220$. Then the number of students who chose soccer and frisbee is $340 - 220 = 120$. Though not required, the total number of students who chose frisbee is $120 + 105 = 225$. Our completed table is shown.

	Soccer	**Basketball**	**Total**
Hockey	220	55	275
Frisbee	120	105	225
Total	340	160	500

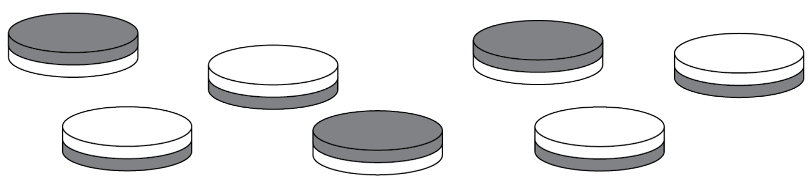
Finally, we can determine that the percentage of students who choose both soccer and frisbee is $\frac{120}{500} \times 100\% = 24\%$.

Problem of the Week
Problem D
Flipping Coins

Yousaf has several plastic coins that are white on one side and grey on the other. He places 16 of these coins on a table with their white sides facing up, and flips them according to the following rules:

1. Flipping a coin changes the colour of the side facing up from either white to grey or from grey to white.
2. Yousaf flips exactly 3 different coins at a time. Each time he does this it is called a *step*.

Determine the fewest steps required until all 16 coins have their grey sides facing up.



Extension:
Suppose Yousaf now starts with 2026 coins on the table with their white sides all facing up. Determine the fewest steps required until all 2026 coins have their grey sides facing up.

Problem of the Week

Problem D and Solution

Flipping Coins

Problem

Yousaf has several plastic coins that are white on one side and grey on the other. He places 16 of these coins on a table with their white sides facing up, and flips them according to the following rules:

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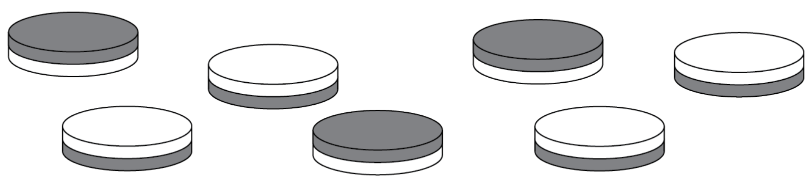


Extension:
Suppose Yousaf now starts with 2026 coins on the table with their white sides all facing up. Determine the fewest steps required until all 2026 coins have their grey sides facing up.

Solution

To simplify the language, a *grey coin* is a coin with its grey side facing up, and a *white coin* is a coin with its white side facing up. We start with 16 white coins and after some number of steps, we want to have 16 grey coins. There are four possible options for each step, assuming there are enough white or grey coins on the table. These options are listed in the following table, along with how each step affects the total number of grey coins:

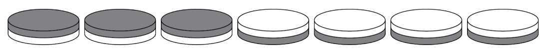
Step Description	Change in Number of Grey Coins
flip 3 white coins to grey	increase by 3
flip 2 white coins to grey and 1 grey coin to white	increase by 1
flip 1 white coin to grey and 2 grey coins to white	decrease by 1
flip 3 grey coins to white	decrease by 3

Since we want the fewest steps possible, we should aim to flip 3 white coins to grey in as many of the steps as we can, as that will increase the total number of grey coins the fastest. If the total number of coins on the table were a multiple of 3, then in every step we could flip 3 white coins to grey. However 16 is not a multiple of 3. Therefore, in at least one of our steps we must *not* flip 3 white coins to grey.

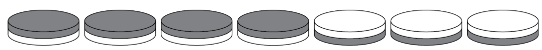
To start with, in each of the first 3 steps we will flip 3 white coins to grey. There will then be $3 \times 3 = 9$ grey coins and $16 - 9 = 7$ white coins.

From here, we will only consider the remaining 7 white coins.

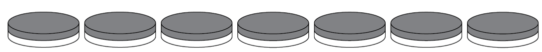
In step 4, we flip 3 white coins to grey. This results in 3 grey coins and 4 white coins, as shown.



In step 5, we flip 2 white coins to grey and 1 grey coin to white. This results in 4 grey coins and 3 white coins, as shown.



Finally, in step 6, we flip 3 white coins to grey. This results in 7 grey coins, as shown.



This method took a total of 6 steps to flip all 16 coins from white to grey. Since only one of these steps did *not* flip 3 white coins to grey, and we determined that at least one of the steps must do this, we could not have achieved our goal in fewer steps.

Therefore, the fewest steps until all 16 coins have their grey sides facing up is 6.

Solution to Extension:

We can use a similar approach for 2026 coins. We notice that $2026 = 3 \times 673 + 7$. Thus, if the first 673 steps are to flip 3 white coins to grey, then we will be left with 7 white coins. We already determined that the fewest steps required to flip 7 white coins to grey is 3. Thus, the fewest steps required until all 2026 coins have their grey sides facing up is $673 + 3 = 676$.

Problem of the Week

Problem D

The Second Escape

Ingrid, Jayant, Neela, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Backwards Thinking, On the Moon, The Family Reunion, or Time Trials. Each of the rooms had a different-coloured door: blue, green, red, or white. While they were in the rooms, each group asked for hints to help them. One group requested 2 hints, another requested 3 hints, another requested 4 hints, and another requested 5 hints.

Use the clues below to match each group with the name of their escape room, the colour of its door, and the number of hints their group requested.

(1) The group that completed The Family Reunion asked for one more hint than Neela's group.

(2) Radko's group went through the green door and requested one fewer hint than the group that completed Backwards Thinking, but one more hint than the group that went through the blue door.

(3) Ingrid's group went through the white door.

(4) The group that completed Time Trials went through the red door and requested two fewer hints than the group that completed On the Moon.

Problem of the Week
Problem D and Solution
The Second Escape

Problem

Ingrid, Jayant, Neela, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Backwards Thinking, On the Moon, The Family Reunion, or Time Trials. Each of the rooms had a different-coloured door: blue, green, red, or white. While they were in the rooms, each group asked for hints to help them. One group requested 2 hints, another requested 3 hints, another requested 4 hints, and another requested 5 hints.

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(1) The group that completed The Family Reunion asked for one more hint than Neela's group.

(2) Radko's group went through the green door and requested one fewer hint than the group that completed Backwards Thinking, but one more hint than the group that went through the blue door.

(3) Ingrid's group went through the white door.

(4) The group that completed Time Trials went through the red door and requested two fewer hints than the group that completed On the Moon.

Solution

From clue (2), we are given information on three different groups. Of these, the group that requested the fewest number of hints went through the blue door. Radko's group requested one more hint and went through the green door. The final group requested one more hint and completed Backwards Thinking. This gives us two possibilities. Either these three groups requested 2, 3, and 4 hints, respectively, or they requested 3, 4, and 5 hints, respectively. We summarize what we know so far for each of these possibilities in the following tables.

Possibility 1:

Number of Hints	2	3	4	5
Group Leader		Radko		
Room Name			Backwards Thinking	
Door Colour	blue	green		

Possibility 2:

Number of Hints	2	3	4	5
Group Leader			Radko	
Room Name				Backwards Thinking
Door Colour		blue	green	

Then, from clue (4) the group that completed Time Trials went through the red door and requested two fewer hints than the group that completed On the Moon. This means the group that completed Time Trials and went through the red door requested either 2 or 3 hints. From this we can eliminate Possibility 1 because neither of the door colours for the rooms that requested 2 or 3 hints can be red. So we will continue with Possibility 2, and conclude that the group that completed Time Trials and went through the red door requested 2 hints. This means that Radko's group completed On the Moon. We summarize what we know so far in the following table.

Number of Hints	2	3	4	5
Group Leader			Radko	
Room Name	Time Trials		On the Moon	Backwards Thinking
Door Colour	red	blue	green	

Then, the group that completed The Family Reunion asked for 3 hints, so from clue (1), Neela's group completed Time Trials. We summarize what we know so far in the following table.

Number of Hints	2	3	4	5
Group Leader	Neela		Radko	
Room Name	Time Trials	The Family Reunion	On the Moon	Backwards Thinking
Door Colour	red	blue	green	

Then, the group that completed Backwards Thinking went through the white door. From clue (3) we can conclude that this was Ingrid's group. Finally, it follows that Jayant's group completed The Family Reunion. The completed puzzle is shown.

Number of Hints	2	3	4	5
Group Leader	Neela	Jayant	Radko	Ingrid
Room Name	Time Trials	The Family Reunion	On the Moon	Backwards Thinking
Door Colour	red	blue	green	white

Data Management (D)

Take me to the cover

Problem of the Week

Problem D

Sports Day

There were 500 students at a sports day camp. For the morning session, each student could choose between soccer or basketball. For the afternoon session, each student could choose between hockey or frisbee. Each student had to choose exactly one sport per session.

The following information is known about the sessions:

- 32% of the students chose basketball,
- $\frac{1}{5}$ of the students who chose hockey also chose basketball, and
- of the students who chose basketball, 50 more chose frisbee than hockey.

Determine the percentage of students who choose both soccer and frisbee.

Problem of the Week

Problem D and Solution

Sports Day

Problem

There were 500 students at a sports day camp. For the morning session, each student could choose between soccer or basketball. For the afternoon session, each student could choose between hockey or frisbee. Each student had to choose exactly one sport per session.

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Determine the percentage of students who choose both soccer and frisbee.

Solution

There were 500 students in total and 32% of them chose basketball. Thus, $0.32 \times 500 = 160$ students chose basketball. Since students had to choose either basketball or soccer, it follows that the remaining $500 - 160 = 340$ students chose soccer. The table shows the information we have determined so far about the number of students who chose each sport.

	Soccer	Basketball	Total
Hockey			
Frisbee			
Total	340	160	500

We know that of the 160 students who chose basketball, 50 more chose frisbee than hockey. So, if h students chose basketball and hockey, then $h + 50$ students chose basketball and frisbee. Since $h + (h + 50) = 160$, it follows that $2h + 50 = 160$, so $2h = 110$, which gives $h = 55$. Thus, 55 of the students who chose basketball also chose hockey. Then the remaining $160 - 55 = 105$ students who chose basketball chose frisbee. We can now update our table accordingly.

	Soccer	Basketball	Total
Hockey		55	
Frisbee		105	
Total	340	160	500

Next, we know that $\frac{1}{5}$ of the students who chose hockey also chose basketball. Since 55 students chose hockey and basketball, this tells us that 55 is $\frac{1}{5}$ of the number of students who chose hockey. Thus, the number of students who chose hockey is $5 \times 55 = 275$. From here we can determine that the number of students who chose soccer and hockey is $275 - 55 = 220$. Then the number of students who chose soccer and frisbee is $340 - 220 = 120$. Though not required, the total number of students who chose frisbee is $120 + 105 = 225$. Our completed table is shown.

	Soccer	**Basketball**	**Total**
Hockey	220	55	275
Frisbee	120	105	225
Total	340	160	500

Finally, we can determine that the percentage of students who choose both soccer and frisbee is $\frac{120}{500} \times 100\% = 24\%$.

Problem of the Week

Problem D

Math Test Math 2

Milana has had six tests in her math class so far. She knows the average (mean) of some of her marks.

- The average of her first and second test marks is 85%.
- The average of her second and third test marks is 82%.
- The average of her third and fourth test marks is 74%.
- The average of her fourth and fifth test marks is 79%.
- The average of her fifth and sixth test marks is 80%.

Determine the average of her first and sixth test marks.

Problem of the Week
Problem D and Solution
Math Test Math 2

Problem
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- The average of her third and fourth test marks is 74%.
- The average of her fourth and fifth test marks is 79%.
- The average of her fifth and sixth test marks is 80%.

Determine the average of her first and sixth test marks.

Solution
Let a, b, c, d, e, and f represent Milana's six test marks, in order. We are looking for the average of a and f. We know the following.

- The average of her first and second test marks is 85, so $\frac{a+b}{2} = 85$, which gives $a + b = 170$.
- The average of her second and third test marks is 82, so $\frac{b+c}{2} = 82$, which gives $b + c = 164$.
- The average of her third and fourth test marks is 74, so $\frac{c+d}{2} = 74$, which gives $c + d = 148$.
- The average of her fourth and fifth test marks is 79, so $\frac{d+e}{2} = 79$, which gives $d + e = 158$.
- The average of her fifth and sixth test marks is 80, so $\frac{e+f}{2} = 80$, which gives $e + f = 160$.

Adding these six equations gives the following:

$$\begin{aligned}
(a+b)+(b+c)+(c+d)+(d+e)+(e+f) &= 170+164+148+158+160 \\
a+2b+2c+2d+2e+f &= 800 \\
a+2(b+c)+2(d+e)+f &= 800 \\
a+2(164)+2(158)+f &= 800 \\
a+328+316+f &= 800 \\
a+f &= 156 \\
\frac{a+f}{2} &= 78
\end{aligned}$$

Therefore, the average of Milana's first and sixth test marks is 78%.

Problem of the Week

Problem D

Come Get Your Tickets!

POTW Secondary School is putting on a play. Tickets for the play are numbered, and each ticket consists of exactly four digits chosen from the digits 0 to 9 (digits may be repeated).

Every possible ticket is printed exactly once. If tickets are handed out in a random order, what is the probability that first ticket handed out has digits whose sum is 34 or higher?

Problem of the Week

Problem D and Solution

Come Get Your Tickets!

Problem

POTW Secondary School is putting on a play. Tickets for the play are numbered, and each ticket consists of exactly four digits chosen from the digits 0 to 9 (digits may be repeated).

Every possible ticket is printed exactly once. If tickets are handed out in a random order, what is the probability that first ticket handed out has digits whose sum is 34 or higher?

Solution

First, we determine the total number of possible four-digit ticket numbers. There are 10 choices for the first digit. For each of these possibilities, there are 10 choices for the second digit. Therefore, there are $10 \times 10 = 100$ possibilities for the first two digits. For each of these possibilities, there are 10 choices for the third digit. Therefore, there are $100 \times 10 = 1000$ possibilities for the first three digits. And finally, for each of these 1000 possibilities for the first three digits, there are 10 choices for the fourth digit. Therefore, there are $1000 \times 10 = 10\,000$ possible four-digit ticket numbers.

Now, we determine the number of ticket numbers with a digit sum of 34 or higher. We consider cases.

- **Case 1:** The ticket number contains four 9s.
  If the digits are all 9s, the sum is 36, which is 34 or higher. There is one ticket with all 9s for digits.

- **Case 2:** The ticket number contains three 9s and one other digit.
  The three 9s sum to 27. In order to get a sum of 34 or higher, the fourth digit must be 7 or 8. Thus, there are two choices for the fourth digit. Once the digit is chosen, there are four places to put the digit. Once the digit is placed, the digits in the remaining spots must be 9s. Therefore, there are $2 \times 4 = 8$ ticket numbers containing three 9s. (It is possible to list them: 7999, 8999, 9799, 9899, 9979, 9989, 9997, 9998.)

- **Case 3:** The ticket number contains two 9s.
  The two 9s sum to 18. In order to get a sum of 34 or higher, we need a sum of $34 - 18 = 16$ or higher from the two remaining digits. The only way to do this, since we cannot use more 9s, is to use two 8s. The two 8s can be placed in six ways and then the 9s must go in the remaining spots. Therefore, there are six ticket numbers containing two 9s. (It is possible to list them: 8899, 8989, 8998, 9889, 9898, 9988.)

- **Case 4:** The ticket number contains one 9.
  In order to get a sum of 34 or higher, we need a sum of $34 - 9 = 25$ or higher from the remaining three digits. But this sum would be made from three digits chosen from the digits 0 to 8. The maximum possible sum would be 24 if three 8s were used. We need the sum to be 25 or higher. Therefore, no ticket number containing only one 9 will produce a sum of 34 or higher.
  It should be noted that a ticket number with zero nines would not produce a sum of 34 or higher either.

Therefore, the number of ticket numbers with a digit sum of 34 or higher is $1 + 8 + 6 = 15$. To calculate the probability we divide the number of tickets with a digit sum of 34 or higher by the number of possible ticket numbers. Therefore, the probability that the first ticket handed out has digits whose sum is 34 or higher is $\frac{15}{10\,000} = \frac{3}{2000}$.

Another way of looking at this result is that out of every 2000 tickets you could expect, on average, 3 tickets with a digit sum of 34 or higher.

Problem of the Week

Problem D

Favourite Number 2

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Ayumi announced that her favourite number is a three-digit positive integer with a digit sum of 9. Vinnie then wrote down a three-digit positive integer with a digit sum of 9. What is the probability that the number Vinnie wrote was Ayumi's favourite number?

Problem of the Week
Problem D and Solution
Favourite Number 2

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Ayumi announced that her favourite number is a three-digit positive integer with a digit sum of 9. Vinnie then wrote down a three-digit positive integer with a digit sum of 9. What is the probability that the number Vinnie wrote was Ayumi's favourite number?

Solution

There are twelve triples of digits that sum to 9. These are: $(9, 0, 0)$, $(8, 1, 0)$, $(7, 2, 0)$, $(7, 1, 1)$, $(6, 3, 0)$, $(6, 2, 1)$, $(5, 4, 0)$, $(5, 3, 1)$, $(5, 2, 2)$, $(4, 4, 1)$, $(4, 3, 2)$, and $(3, 3, 3)$.

We will count the number of three-digit positive integers whose digit sum is 9 by counting the number of positive integers formed by the digits in each triple above. We will consider cases based on the number of zeroes in each triple, since a three-digit integer cannot have a leading digit of 0.

- **Case 1:** Two zeroes
  The only triple with two zeros is $(9, 0, 0)$. Since the first digit of the three-digit integer cannot be zero, the only possible three-digit integer is 900. Thus, there is only 1 possible three-digit integer with two zeroes.

- **Case 2:** One zero
  The triples with one zero are $(8, 1, 0)$, $(7, 2, 0)$, $(6, 3, 0)$, and $(5, 4, 0)$. Consider $(8, 1, 0)$. Since the zero cannot be in the first position, there are four integers that can be formed with these three digits: 810, 801, 180, and 108. Since this is true for any of the four triples in this case, there are $4 \times 4 = 16$ possible three-digit integers with one zero.

- **Case 3:** No zeroes
  In this case, we will further separate the triples based on the number of repeated digits.

  - **Case 3a:** Three repeated digits
    The only triple with three repeated digits is $(3, 3, 3)$ and the only integer formed by these digits is 333. Thus, there is only 1 possible three-digit integer in this case.

- **Case 3b:** Two repeated digits
  The triples with two repeated digits are $(7, 1, 1)$, $(5, 2, 2)$, and $(4, 4, 1)$. Consider $(7, 1, 1)$. There are three integers that can be formed by these three digits: 711, 171, and 117. Since this is true for any of the three triples in this case, there are $3 \times 3 = 9$ possible three-digit integers in this case.
- **Case 3c:** No repeated digits
  The triples are $(6, 2, 1)$, $(5, 3, 1)$, and $(4, 3, 2)$. Consider $(6, 2, 1)$. There are six integers that can be formed by these three digits: 621, 612, 261, 216, 162, and 126. Since this is true for any of the three triples in this case, there are $3 \times 6 = 18$ possible three-digit integers in this case.

Therefore, there are $1 + 9 + 18 = 28$ possible three-digit integers with no zeroes.

Therefore, the total number of three-digit positive integers that have a digit sum of 9 is $1 + 16 + 28 = 45$. Since Vinnie wrote down only one of these integers, the probability that this integer was Ayumi's favourite number is $\frac{1}{45}$.

Note: It is a known fact that an integer is divisible by 9 exactly when its digit sum is divisible by 9. For example, 32 814 has a digit sum of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 9, then 32 814 is divisible by 9. On the other hand, 32 810 has a digit sum of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 9, then 32 810 is not divisible by 9.

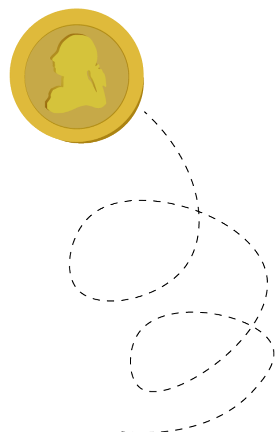
As a consequence of this fact, Ayumi's favourite number must be divisible by 9.

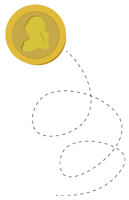
Problem of the Week

Problem D

Coin Game

Subhash and Marijn take turns tossing a fair coin. Subhash goes first and each player has a total of two turns. The first player to toss a tail wins. If neither player tosses a tail, then neither player wins. What is the probability that Subhash wins?





Problem of the Week

Problem D and Solution

Coin Game

Problem

Subhash and Marijn take turns tossing a fair coin. Subhash goes first and each player has a total of two turns. The first player to toss a tail wins. If neither player tosses a tail, then neither player wins. What is the probability that Subhash wins?

Solution

Subhash can win on either his first or second turn. We will calculate the probability for each case. Note that the probability of a specific toss on any turn is $\frac{1}{2}$, since the coin is fair.

- **Case 1:** Subhash wins on his first turn.
  If Subhash wins on his first turn, then he must have tossed tails on this turn. The probability of this is $\frac{1}{2}$.

- **Case 2:** Subhash wins on his second turn.
  If Subhash wins on his second turn, then he must have tossed heads on his first turn, then Marijn must have tossed heads, and then Subhash must have tossed tails. We will call this sequence HHT. Note that this sequence is the only sequence that allows Subhash to win on his second turn. The probability of this sequence of tosses is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Therefore, the probability that Subhash wins is $\frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8} = 62.5\%$.

Geometry & Measurement (G)

Take me to the cover

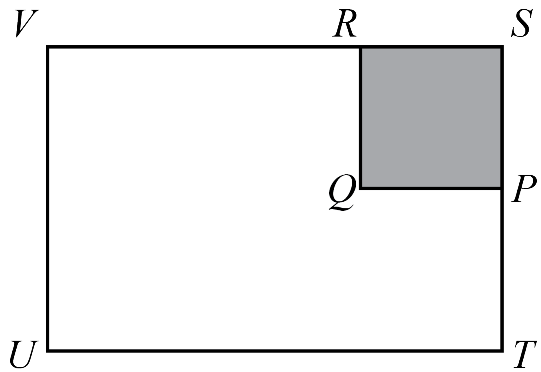
Problem of the Week

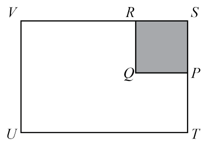
Problem D

Cutting the Corner

Rectangle $STUV$ has P on ST, R on SV, and Q inside the rectangle such that $PQRS$ is a square. When square $PQRS$ is removed from rectangle $STUV$, the remaining shape has an area of 92 m^2.

If $PT = 4$ m and $RV = 8$ m, what is the area of rectangle $STUV$?





Problem of the Week

Problem D and Solution

Cutting the Corner

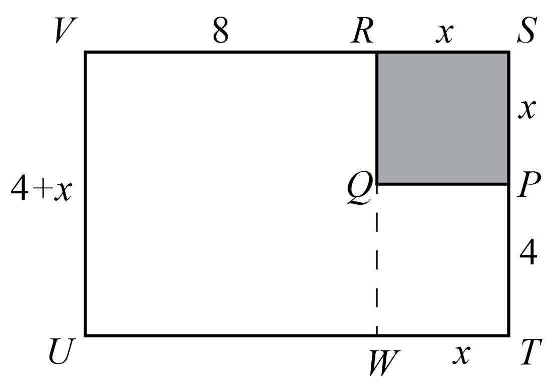
Problem

Rectangle $STUV$ has P on ST, R on SV, and Q inside the rectangle such that $PQRS$ is a square. When square $PQRS$ is removed from rectangle $STUV$, the remaining shape has an area of 92 m^2.

If $PT = 4$ m and $RV = 8$ m, what is the area of rectangle $STUV$?

Solution

Let x represent the side length of square $PQRS$. In the diagram, extend RQ to intersect TU at W. This creates rectangle $PTWQ$ and rectangle $RWUV$. Also, $UV = PT + SP = (4 + x)$ m and $TW = RS = x$ m.



We are given that area $PTWQ$ + area $RWUV = 92$ m^2. That is,

$$\begin{aligned} PT \times TW + RV \times UV &= 92 \\ 4x + 8(4 + x) &= 92 \\ 4x + 32 + 8x &= 92 \\ 12x + 32 &= 92 \\ 12x &= 60 \\ x &= 5 \end{aligned}$$

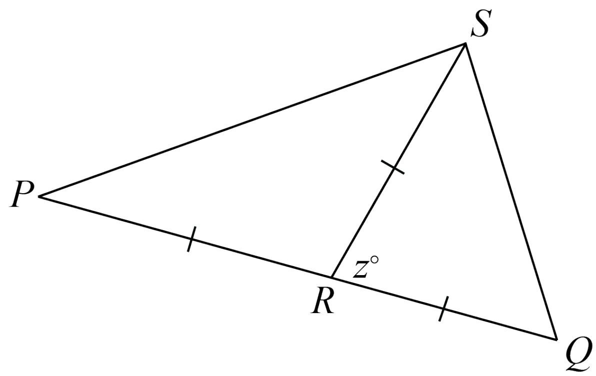
Thus, $x = 5$ m, and so $SV = 8 + x = 13$ m, and $UV = 4 + x = 9$ m.

Therefore, the area of rectangle $STUV$ is $SV \times UV = 13 \times 9 = 117$ m^2.

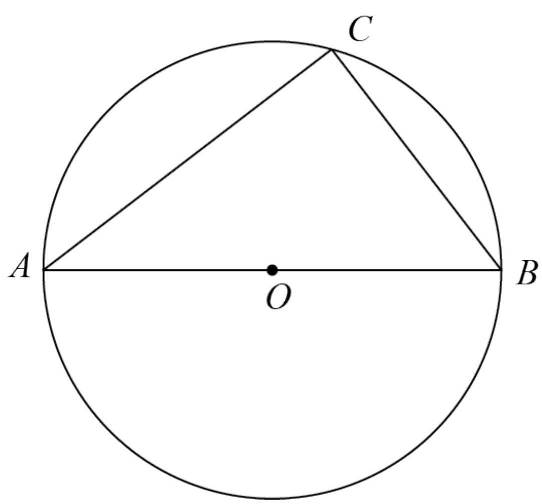
Problem of the Week
Problem D
Triangle Chase

In $\triangle PQS$, R lies on PQ such that $PR = RQ = RS$ and $\angle QRS = z^\circ$.

Determine the measure of $\angle PSQ$.



Extension: Consider the circle with centre O and diameter AB. Suppose point C is a point on the circumference of the circle other than A or B. Determine the measure of $\angle ACB$.



Problem of the Week
Problem D and Solution
Triangle Chase

Problem

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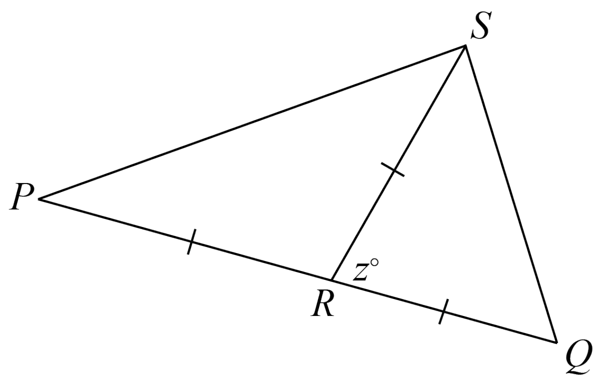

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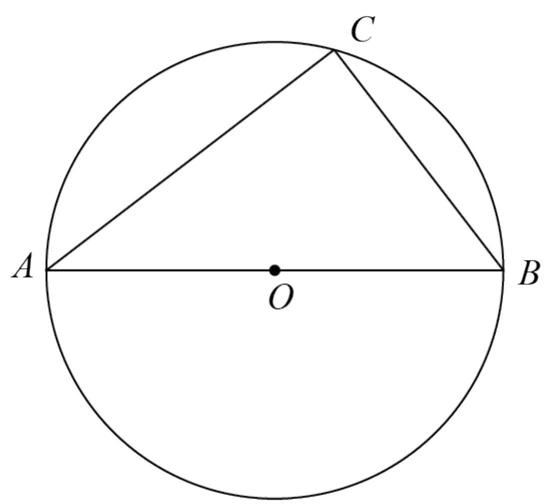
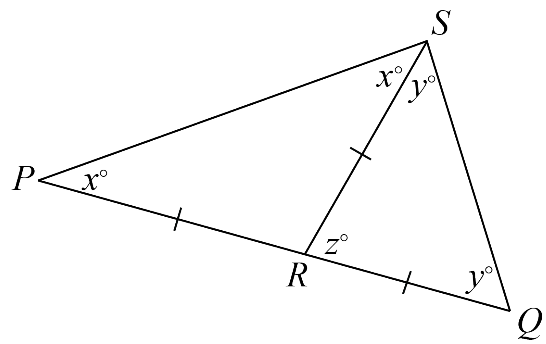


Solution

Let $\angle RPS = x^\circ$ and $\angle RQS = y^\circ$.

In $\triangle PRS$, since $PR = RS$, $\triangle PRS$ is isosceles and $\angle RSP = \angle RPS = x^\circ$. Similarly, in $\triangle QRS$, since $RQ = RS$, $\triangle QRS$ is isosceles and $\angle RSQ = \angle RQS = y^\circ$.



From here we proceed with two different solutions.

Solution 1

Since PRQ is a straight line, $\angle PRS + \angle QRS = 180^\circ$. Since $\angle QRS = z^\circ$, we have $\angle PRS = 180 - z^\circ$.

The angles in a triangle sum to 180°, so in $\triangle PRS$

$$\begin{aligned}\angle RPS + \angle RSP + \angle PRS &= 180^\circ\\ x^\circ + x^\circ + 180^\circ - z^\circ &= 180^\circ\\ 2x &= z\\ x &= \frac{z}{2}\end{aligned}$$

The angles in a triangle sum to 180°, so in $\triangle QRS$

$$\begin{aligned}\angle RQS + \angle RSQ + \angle QRS &= 180^\circ\\ y^\circ + y^\circ + z^\circ &= 180^\circ\\ 2y &= 180 - z\\ y &= \frac{180 - z}{2}\end{aligned}$$

Then $\angle PSQ = \angle RSP + \angle RSQ = x^\circ + y^\circ = \frac{z}{2}^\circ + \left(\frac{180-z}{2}\right)^\circ = \left(\frac{180}{2}\right)^\circ = 90^\circ$.

Therefore, the measure of $\angle PSQ$ is 90°.

It turns out that it is not necessary to determine expressions for x and y in terms of z to solve this problem, as we'll see in Solution 2.

Solution 2

The angles in a triangle sum to 180°, so in $\triangle PQS$

$$\begin{aligned}\angle QPS + \angle PSQ + \angle PQS &= 180^\circ\\ x^\circ + (x^\circ + y^\circ) + y^\circ &= 180^\circ\\ (x^\circ + y^\circ) + (x^\circ + y^\circ) &= 180^\circ\\ 2(x^\circ + y^\circ) &= 180^\circ\\ x^\circ + y^\circ &= 90^\circ\end{aligned}$$

But $\angle PSQ = \angle RSP + \angle RSQ = x^\circ + y^\circ = 90^\circ$.

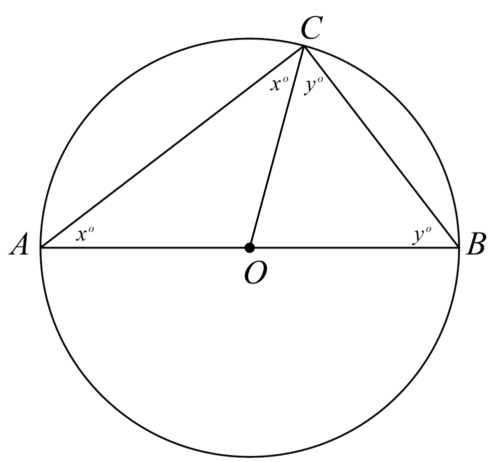
Therefore, the measure of $\angle PSQ$ is 90°.

Extension Solution:

Join C to the centre O. Since OA, OB and OC are radii of the circle, $OA = OB = OC$. Let $\angle CAO = x^\circ$ and $\angle CBO = y^\circ$.

Since $OA = OC$, $\triangle AOC$ is isosceles and $\angle ACO = \angle CAO = x^\circ$.

Since $OB = OC$, $\triangle BOC$ is isosceles and $\angle BCO = \angle CBO = y^\circ$.



The angles in a triangle add to 180°, so in $\triangle ABC$

$$\begin{aligned}\angle ACB + \angle CAB + \angle CBA &= 180^\circ\\ (x^\circ + y^\circ) + x^\circ + y^\circ &= 180^\circ\\ 2(x^\circ + y^\circ) &= 180^\circ\\ x^\circ + y^\circ &= 90^\circ\end{aligned}$$

But $\angle ACB = x^\circ + y^\circ$, so $\angle ACB = 90^\circ$.

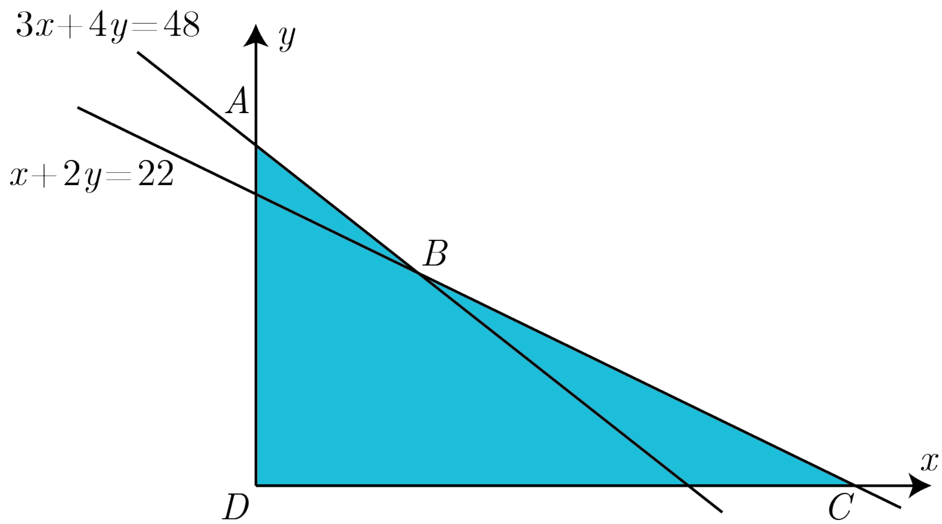
This result is often expressed as a theorem for circles: *An angle inscribed in a circle by a diameter of the circle is* 90°.

Problem of the Week

Problem D

Shading the Quad

Peggie draws the line $3x + 4y = 48$ and labels its intersection with the y-axis A. Derrick then draws the line $x + 2y = 22$ on the same set of axes and labels its intersection with Peggie's line B. Derrick then labels the intersection of his line with the x-axis C. Peggie then labels the origin D and shades in quadrilateral $ABCD$, as shown.

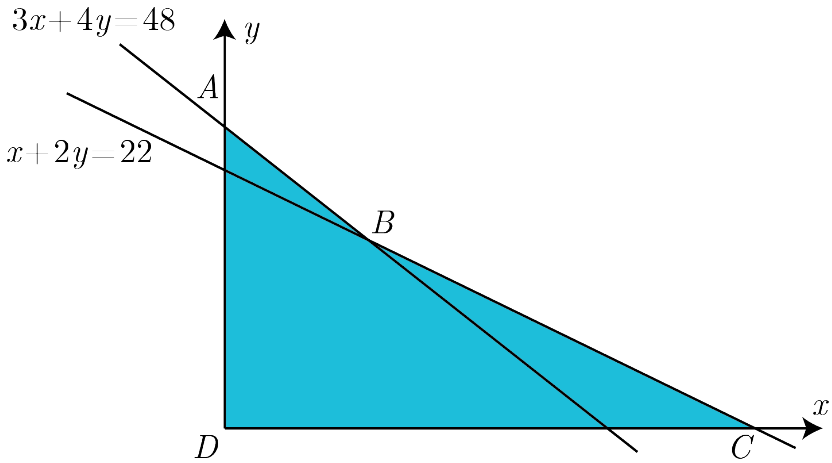


Determine the area of quadrilateral $ABCD$.

Problem of the Week
Problem D and Solution
Shading the Quad

Problem

Peggie draws the line $3x + 4y = 48$ and labels its intersection with the y-axis A. Derrick then draws the line $x + 2y = 22$ on the same set of axes and labels its intersection with Peggie's line B. Derrick then labels the intersection of his line with the x-axis C. Peggie then labels the origin D and shades in quadrilateral $ABCD$, as shown.



Determine the area of quadrilateral $ABCD$.

Solution

We start by finding the coordinates of points A, B, and C. Since D is the origin, its coordinates are $(0, 0)$.

To determine the coordinates of point A, we need to find the y-intercept of the line $3x + 4y = 48$. To find this, let $x = 0$. Then $4y = 48$, so $y = 12$. Thus, the coordinates of A are $(0, 12)$.

To determine the coordinates of point C, we need to find the x-intercept of the line $x + 2y = 22$. To find this, let $y = 0$. Then $x = 22$. Thus, the coordinates of C are $(22, 0)$.

Point B is the point of intersection of the lines $3x + 4y = 48$ and $x + 2y = 22$. To find this, we solve the following system of equations.

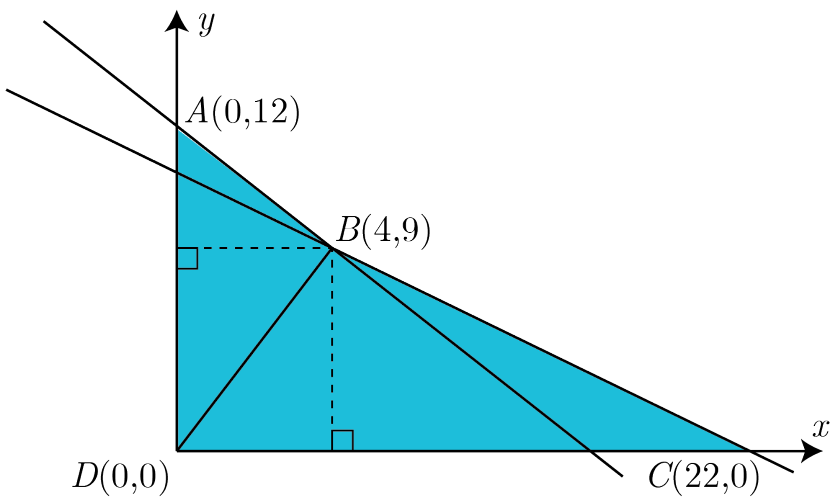
$$3x + 4y = 48 \qquad (1)$$
$$x + 2y = 22 \qquad (2)$$

From equation (2), we solve for x to obtain $x = 22 - 2y$. Substituting into equation (1), we have

$$\begin{aligned} 3(22 - 2y) + 4y &= 48 \\ 66 - 6y + 4y &= 48 \\ -2y &= -18 \\ y &= 9 \end{aligned}$$

Then substituting back into $x = 22 - 2y$, we obtain $x = 22 - 2(9) = 4$. Thus, the coordinates of B are $(4, 9)$.

We now draw line segment BD to divide quadrilateral $ABCD$ into two triangles: $\triangle ABD$ and $\triangle CBD$ as shown.



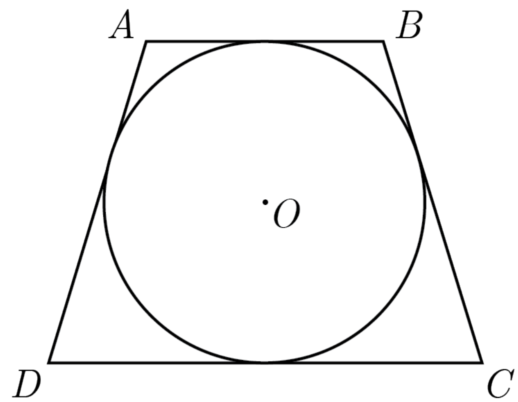
$\triangle ABD$ has a base of 12 and a height of 4, and $\triangle CBD$ has a base of 22 and a height of 9. Thus,

$$\begin{aligned} \text{Area } ABCD &= \text{Area } \triangle ABD + \text{Area } \triangle CBD \\ &= \frac{1}{2}(12)(4) + \frac{1}{2}(22)(9) \\ &= 24 + 99 \\ &= 123 \end{aligned}$$

Therefore, the area of quadrilateral $ABCD$ is 123 units^2.

Problem of the Week
Problem D
A Circle in a Trapezoid

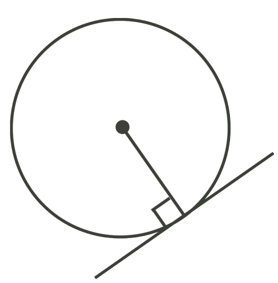
A circle with centre O and radius 15 m is inside trapezoid $ABCD$ such that each side of $ABCD$ is tangent to the circle. In the trapezoid, $AB \parallel CD$ and $AD = BC$, so $ABCD$ is an isosceles trapezoid.



If the area of $ABCD$ is 2025 m^2, determine the lengths of AD and BC.

Note: For this problem, you may want to use the following known results about circles:

1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
2. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.



Problem of the Week

Problem D and Solution

A Circle in a Trapezoid

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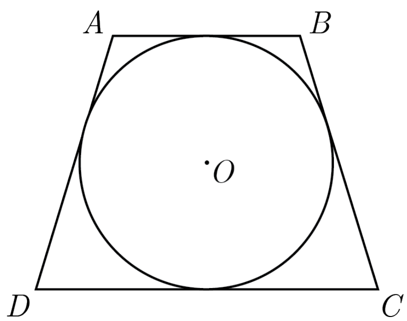

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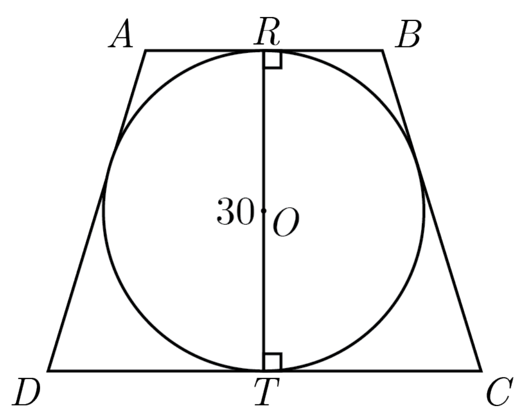
1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
2. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.

Solution

Draw line segment RT through O perpendicular to AB and CD so that R lies on AB and T lies on CD. From Result 2 given after the problem, R and T are the points of tangency on AB and CD, respectively.

Since OR and OT are both radii of the circle, $RT = OR + OT = 15 + 15 = 30$. Since $RT \perp CD$, RT is the height of trapezoid $ABCD$.



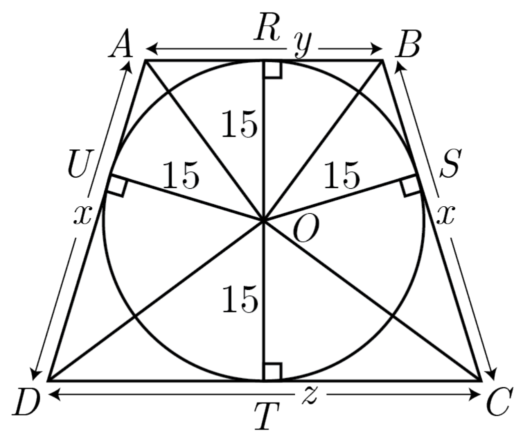
We will now proceed with two different approaches.

Solution 1

Let $y = AB$ and $z = CD$. Then using the formula for the area of a trapezoid,

$$\begin{aligned}\text{Area} &= (AB + CD) \times (RT) \div 2 \\ 2025 &= (y + z) \times 30 \div 2 \\ 2025 &= (y + z) \times 15 \\ 135 &= y + z \qquad (1)\end{aligned}$$

Let $x = AD = BC$. Join O to each of the vertices of $ABCD$, creating four triangles, $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$. Then join O to each of the points of tangency, R, S, T, and U, on AB, BC, CD, and AD, respectively. Each of these line segments is a radius so $OR = OS = OT = OU = 15$. Furthermore, from Result 1 given after the problem, these line segments are each perpendicular to their respective sides of the trapezoid.



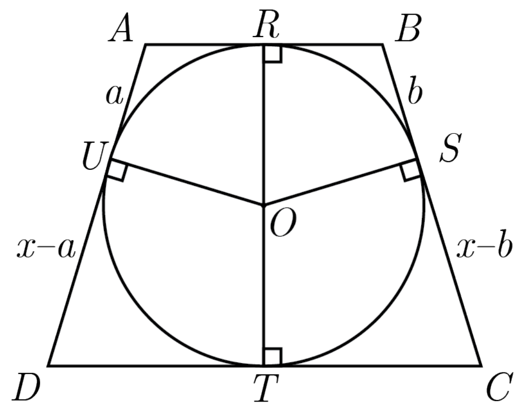
We can now find the area of the trapezoid a second way by summing the areas of the four triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$.

$$\begin{aligned}\text{Area} &= \text{Area } \triangle AOB + \text{Area } \triangle BOC + \text{Area } \triangle COD + \text{Area } \triangle AOD \\ 2025 &= \frac{AB \times OR}{2} + \frac{BC \times OS}{2} + \frac{CD \times OT}{2} + \frac{AD \times OU}{2} \\ 2025 &= \frac{15y}{2} + \frac{15x}{2} + \frac{15z}{2} + \frac{15x}{2} \\ 2025 &= 15x + \frac{15(y+z)}{2} \\ 2025 &= 15x + \frac{15(135)}{2} \qquad \text{(from (1))} \\ 2025 &= 15x + \frac{2025}{2} \\ 15x &= \frac{2025}{2} \\ x &= \frac{135}{2} = 67.5\end{aligned}$$

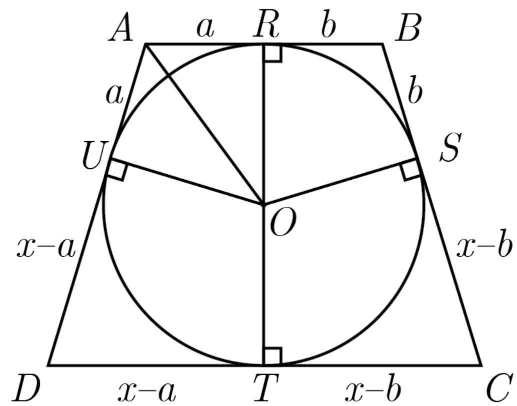
Therefore, AD and BC each have a length of 67.5 m.

Solution 2

Join O to each of the points of tangency, R, S, T, and U, on AB, BC, CD, and AD, respectively. From Result 1 given after the problem, these line segments are each perpendicular to their respective sides of the trapezoid.
Let $x = AD = BC$, $a = AU$ and $b = BS$. Then $DU = x - a$ and $CS = x - b$.



Join O to A, forming right-angled triangles $\triangle AUO$ and $\triangle ARO$. Using the Pythagorean theorem, $AO^2 = AU^2 + OU^2$ and $AO^2 = AR^2 + OR^2$. Since OU and OR are both radii, $OU = OR$. Thus $AR = AU = a$. Using a similar reasoning, we can conclude that $BR = BS = b$, $CT = CS = x - b$, and $DT = DU = x - a$.



Then using the formula for the area of a trapezoid,

$$\begin{aligned}
\text{Area} &= (AB + CD) \times (RT) \div 2 \\
2025 &= ((AR + BR) + (DT + CT)) \times 30 \div 2 \\
2025 &= ((a + b) + (x - a + x - b) \times 15 \\
2025 &= 2x \times 15 \\
2025 &= 30x \\
x &= \frac{2025}{30} = 67.5
\end{aligned}$$

Therefore, AD and BC each have a length of 67.5 m.

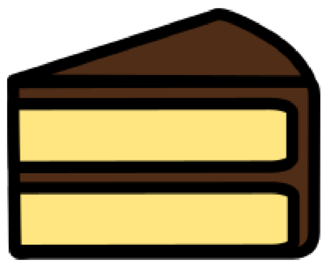
Problem of the Week

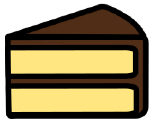
Problem D

Cheesecake Geometry

For Amanda's birthday, Rhett made a chocolate cheesecake. The cake was in the shape of a cylinder, with equal radius and height. Rhett cut the cake into 8 congruent slices, each in the shape of a sector of a cylinder. Rhett then ate one slice for quality control purposes.

After removing Rhett's slice, by what percentage has the surface area of the cake increased or decreased? Round your answer to one decimal place.





Problem of the Week

Problem D and Solution

Cheesecake Geometry

Problem

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After removing Rhett's slice, by what percentage has the surface area of the cake increased or decreased? Round your answer to one decimal place.

Solution

Let the cake have radius r and height h. We are given that $r = h$.

We first calculate the surface area of the entire cake without the slice removed.

The surface area of a cylinder includes the areas of the two circular faces plus the surface area of the side. The circular faces each have area equal to πr^2. Since the circumference of a circular face is $2\pi r$, the surface area of the side is $(2\pi r)(h) = 2\pi r^2$, since $r = h$. Thus, the total surface area of the cake before the slice is removed is $\pi r^2 + \pi r^2 + 2\pi r^2 = 4\pi r^2$.

We now calculate the surface area of the cake after the slice is removed.

The surface area removed includes $\frac{1}{8}$ of the area of each of the two circular faces with radius r and $\frac{1}{8}$ of the surface area of the side of the cylinder. Therefore, the surface area remaining includes $1 - \frac{1}{8} = \frac{7}{8}$ of the original surface area. That is, $\frac{7}{8} \times 4\pi r^2 = \frac{7\pi r^2}{2}$.

There are two areas that are added as a result of removing the slice. These areas are the rectangular faces of the slice that were inside the cylindrical cake before the slice was removed. These rectangles each have length equal to the radius of the circular face, r, and width $h = r$. Thus, the surface area added is equal to $2(r)(r) = 2r^2$.

Therefore, the new total surface area is equal to $\frac{7\pi r^2}{2} + 2r^2$.

The change in surface area is $(\frac{7\pi r^2}{2} + 2r^2) - 4\pi r^2 = -\frac{\pi r^2}{2} + 2r^2 = r^2(2 - \frac{\pi}{2})$.

Now, $\frac{\pi}{2} < 2$, so $2 - \frac{\pi}{2} > 0$. Therefore, the surface area increases as a result of removing the slice.

To calculate the percentage that the area has increased, we divide the surface area increase by the original surface area, and multiply by 100%.

The increase is $r^2\left(2 - \frac{\pi}{2}\right)$. Therefore, the percentage increase in surface area is equal to

$$\begin{aligned}\frac{r^2\left(2 - \frac{\pi}{2}\right)}{4\pi r^2} \times 100\% &= \frac{\left(2 - \frac{\pi}{2}\right)}{4\pi} \times 100\% \\ &= \left(\frac{4 - \pi}{8\pi}\right) \times 100\% \\ &\approx 3.4\%\end{aligned}$$

Therefore, the surface area of the cake increases by approximately 3.4% after the slice has been removed.

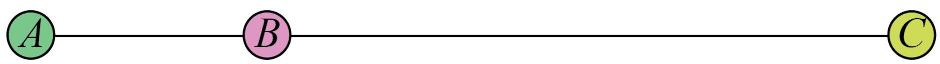
Problem of the Week

Problem D

Walking Away

Anna, Biljana, and Cees stand in a row in a large field. Anna is 180 m west of Biljana and Cees is 450 m east of Biljana. At the same time, Anna and Cees begin walking. Anna walks north at a constant rate of 75 m/min and Cees walks south at a constant rate of 90 m/min. Biljana does not move.

After how many minutes of walking will the distance between Biljana and Cees be twice the distance between Anna and Biljana?



Problem of the Week
Problem D and Solution
Walking Away

Problem

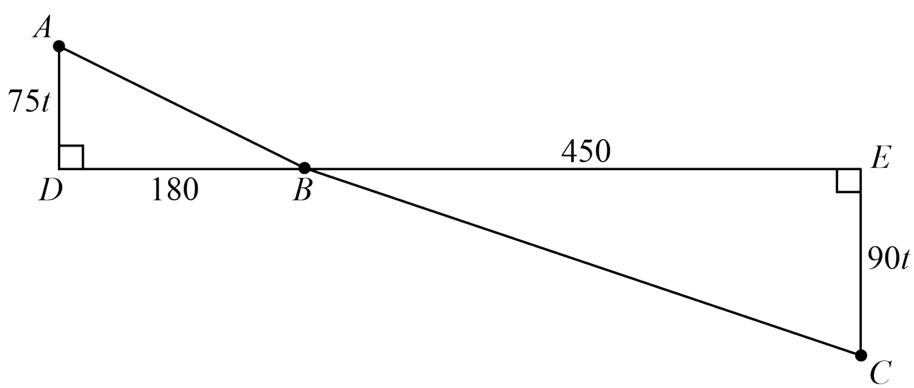
Anna, Biljana, and Cees stand in a row in a large field. Anna is 180 m west of Biljana and Cees is 450 m east of Biljana. At the same time, Anna and Cees begin walking. Anna walks north at a constant rate of 75 m/min and Cees walks south at a constant rate of 90 m/min. Biljana does not move.

After how many minutes of walking will the distance between Biljana and Cees be twice the distance between Anna and Biljana?

Solution

Solution 1

Let t represent the number of minutes until the distance between Biljana and Cees is twice the distance between Anna and Biljana. In t minutes Anna will walk $75t$ m north and Cees will walk $90t$ m south. The following diagram shows Anna's position, A, Biljana's position, B, and Cees's position, C, in metres, at time $t > 0$.



Since both triangles in the diagram are right-angled triangles, we can use the Pythagorean Theorem to determine the value of t when $BC = 2AB$.

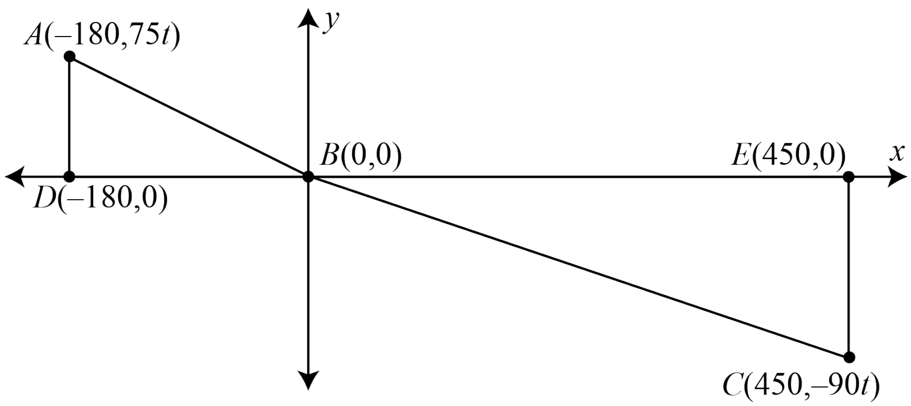
$$\begin{aligned}
BC &= 2AB \\
(BC)^2 &= (2AB)^2 \\
BE^2 + EC^2 &= 4(AD^2 + DB^2) \\
450^2 + (90t)^2 &= 4((75t)^2 + 180^2) \\
202\,500 + 8100t^2 &= 4(5625t^2 + 32\,400) \\
202\,500 + 8100t^2 &= 22\,500t^2 + 129\,600 \\
72\,900 &= 14\,400t^2 \\
\frac{81}{16} &= t^2
\end{aligned}$$

Since $t > 0$, it follows that $t = \frac{9}{4} = 2\frac{1}{4}$ min. Therefore, after $2\frac{1}{4}$ minutes of walking, the distance between Biljana and Cees will be twice the distance between Anna and Biljana.

Solution 2

In this solution, we represent Anna, Biljana and Cees's original positions as points on the x-axis so that Biljana is positioned at the origin $B(0,0)$, Anna is positioned 180 units left of Biljana at $D(-180,0)$ and Cees is positioned 450 units right of Biljana at $E(450,0)$.

Let t represent the number of minutes until the distance between Biljana and Cees is twice the distance between Anna and Biljana. In t minutes Anna will walk $75t$ m north to the point $A(-180,75t)$, and Cees will walk $90t$ m south to the point $C(450,-90t)$.



The distance from a point $P(x,y)$ to the origin can be found using the formula $d=\sqrt{x^2+y^2}$.

Then $AB=\sqrt{(-180)^2+(75t)^2}=\sqrt{32\,400+5625t^2}$ and
$BC=\sqrt{(450)^2+(-90t)^2}=\sqrt{202\,500+8100t^2}$.

$$\begin{aligned}
BC &= 2AB\\
\sqrt{202\,500+8100t^2} &= 2\sqrt{32\,400+5625t^2}\\
(\sqrt{202\,500+8100t^2})^2 &= (2\sqrt{32\,400+5625t^2})^2\\
202\,500+8100t^2 &= 4(32\,400+5625t^2)\\
202\,500+8100t^2 &= 129\,600+22\,500t^2\\
72\,900 &= 14\,400t^2\\
\frac{81}{16} &= t^2
\end{aligned}$$

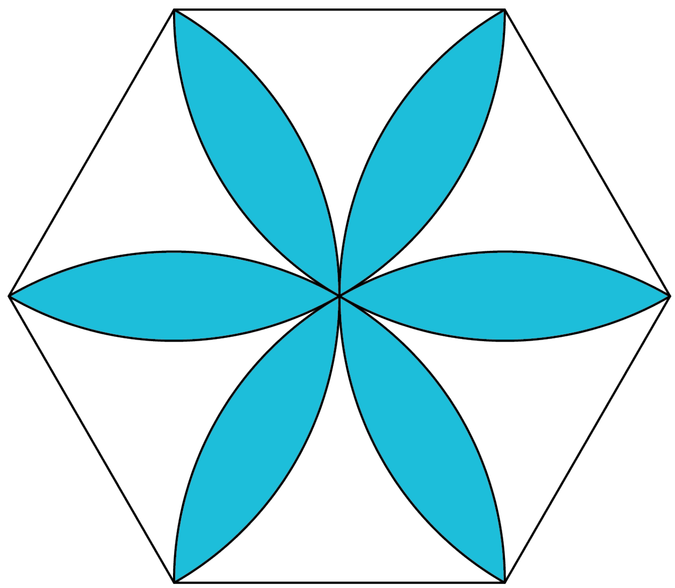
Since $t>0$, it follows that $t=\frac{9}{4}=2\frac{1}{4}$ min. Therefore, after $2\frac{1}{4}$ minutes of walking, the distance between Biljana and Cees will be twice the distance between Anna and Biljana.

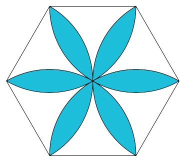
Problem of the Week

Problem D

A New Pi Plate

A pi-plate is in the shape of a regular hexagon with side length 8, and has an interesting pattern on it. From each vertex in the hexagon, a circular arc with centre at the vertex and radius 8 is drawn. This results in a flower-like pattern inside the hexagon, which is then shaded. Determine the total area of the shaded region.





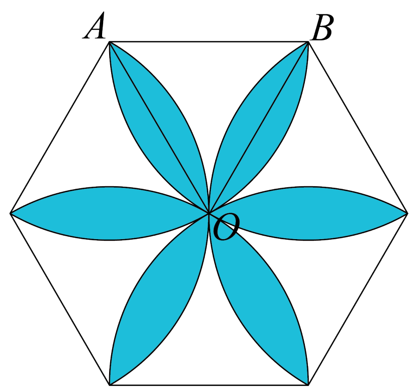
Problem of the Week
Problem D and Solution
A New Pi Plate

Problem

A pi-plate is in the shape of a regular hexagon with side length 8, and has an interesting pattern on it. From each vertex in the hexagon, a circular arc with centre at the vertex and radius 8 is drawn. This results in a flower-like pattern inside the hexagon, which is then shaded. Determine the total area of the shaded region.

Solution

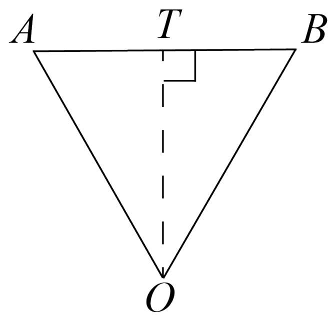
We label the centre of the hexagon O and consider two adjacent vertices, A and B. We draw in OA and OB.



We note that a regular hexagon is composed of six congruent equilateral triangles (the proof of this is left up to the student). Thus, $\triangle AOB$ is an equilateral triangle with side length 8.

To determine the total shaded area, we can consider the area of one-half of each 'petal' of the flower, and then multiply this area by 12, since there are 12 half-petals in the diagram. To determine the area of the region between the arc through B and O and the line segment OB, we can calculate the area of the sector AOB, with centre A and radius AB, and subtract the area of $\triangle AOB$.

We first determine the area of $\triangle AOB$. Construct altitude OT. Since $\triangle AOB$ is equilateral, it follows that OT bisects AB.



Since $AB = 8$, it follows that $AT = 4$. By the Pythagorean Theorem in $\triangle ATO$, $AO^2 = AT^2 + OT^2$. Therefore, $8^2 = 4^2 + OT^2$, and $OT^2 = 64 - 16 = 48$ follows. Since $OT > 0$, we have $OT = \sqrt{48}$.

Therefore, the area of $\triangle AOB$ is $\dfrac{(AB) \times (OT)}{2} = \dfrac{8 \times \sqrt{48}}{2} = 4\sqrt{48} = 16\sqrt{3}$.

We now determine the area of the sector AOB. Since $\angle OAB = 60^\circ$, the area of sector AOB is $\frac{1}{6}$ of the area of a circle with radius 8. Thus, the area of the sector is equal to $\frac{1}{6}\pi(AB)^2 = \frac{1}{6}\pi(8)^2 = \frac{32}{3}\pi$.

Therefore, the area of the shaded region between the arc through B and O and the line segment OB is $\left(\frac{32}{3}\pi - 16\sqrt{3}\right)$ units^2.

The original pi-plate has 12 of these regions, so the entire shaded area is $12 \times (\frac{32}{3}\pi - 16\sqrt{3}) = 128\pi - 192\sqrt{3} \approx 69.57$ units^2.

Problem of the Week

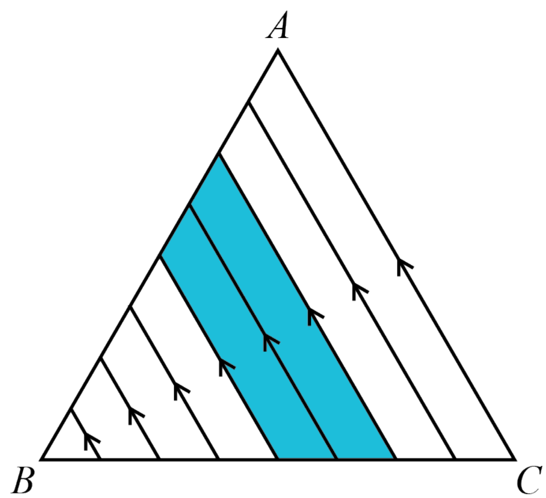
Problem D

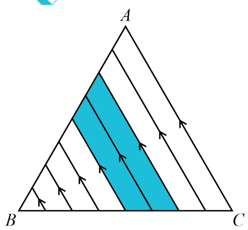
Another Suncatcher

A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

- Sides AB and BC are each divided into 8 segments of equal length.
- Each point of division on AB is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

Two of the sections are coloured blue to form a trapezoid, as shown. Determine the area of this trapezoid.





Problem of the Week
Problem D and Solution
Another Suncatcher

Problem

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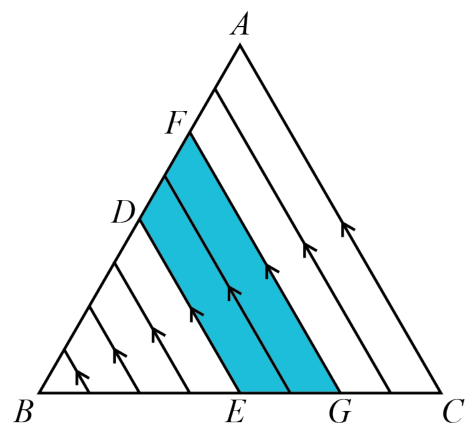
Two of the sections are coloured blue to form a trapezoid, as shown. Determine the area of this trapezoid.

Solution

Solution 1

We start by labeling the vertices of the trapezoid D, E, F, and G, as shown. In this solution we will subtract the area of $\triangle BDE$ from the area of $\triangle BFG$ to find the area of trapezoid $DEGF$.



Since AB and BC are each divided into 8 equal segments, each of these segments must have length $144 \div 8 = 18$ mm. Since BD is made up of 4 of these segments, $BD = 4 \times 18 = 72$ mm. Similarly, since BF is made up of 6 of these segments, $BF = 6 \times 18 = 108$ mm.

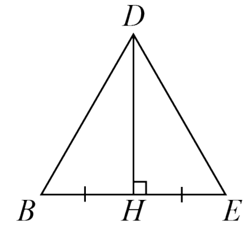
First we will show that $\triangle BDE$ and $\triangle BFG$ are equilateral triangles. Since $\triangle ABC$ is equilateral, $\angle BAC = \angle ABC = \angle ACB = 60^\circ$. Since $\angle DBE$, $\angle FBG$ and $\angle ABC$ are the same angle, $\angle DBE = \angle FBG = \angle ABC = 60^\circ$.

Since $DE \parallel FG \parallel AC$, $\angle BED = \angle BGF = \angle BCA = 60^\circ$ and $\angle BDE = \angle BFG = \angle BAC = 60^\circ$.

Since each angle in $\triangle BDE$ and $\triangle BFG$ is 60°, both triangles are equilateral. $\triangle BDE$ has side length 72 mm and $\triangle BFG$ has side length 108 mm.

Next we need to determine the height of each triangle. In $\triangle BDE$, drop a perpendicular from D to H on BE. Since $\triangle BDE$ is equilateral, H is the midpoint of BE and it follows that $BH = \frac{1}{2}BE = \frac{1}{2}(72) = 36$ mm.

Using the Pythagorean Theorem in $\triangle BHD$,



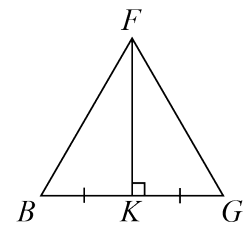
$$\begin{aligned} DH^2 &= BD^2 - BH^2 \\ &= 72^2 - 36^2 = 3888 \end{aligned}$$

Therefore $DH = \sqrt{3888} = 36\sqrt{3}$ mm, since $DH \geq 0$.

Therefore, the area of $\triangle BDE$ is $\frac{(BE)(DH)}{2} = \frac{(72)(36\sqrt{3})}{2} = 1296\sqrt{3}$ mm^2.

In $\triangle BFG$, drop a perpendicular from F to K on BG. Since $\triangle BFG$ is equilateral, K is the midpoint of BG and it follows that $BK = \frac{1}{2}BG = \frac{1}{2}(108) = 54$ mm.

Using the Pythagorean Theorem in $\triangle BKF$,



$$\begin{aligned} FK^2 &= BF^2 - BK^2 \\ &= 108^2 - 54^2 = 8748 \end{aligned}$$

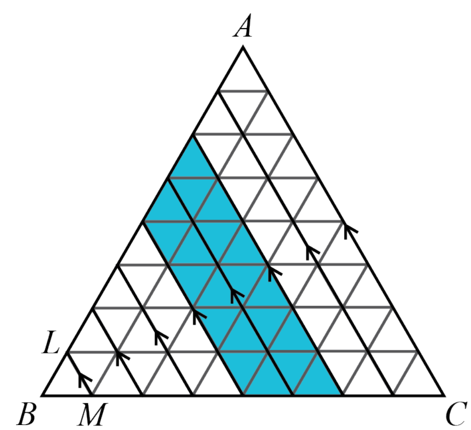
Therefore $FK = \sqrt{8748} = 54\sqrt{3}$ mm, since $FK \geq 0$.

Therefore, the area of $\triangle BFG$ is $\frac{(BG)(FK)}{2} = \frac{(108)(54\sqrt{3})}{2} = 2916\sqrt{3}$ mm^2. Thus, the area of trapezoid $DEGF$ is area $\triangle BFG -$ area $\triangle BDE = 2916\sqrt{3} - 1296\sqrt{3} = 1620\sqrt{3}$ mm^2.

NOTE: We could also have found the lengths of DH and FK by recognizing that $\triangle BHD$ and $\triangle BKF$ are $30° - 60° - 90°$ triangles with side lengths in the ratio $1 : \sqrt{3} : 2$.

Solution 2

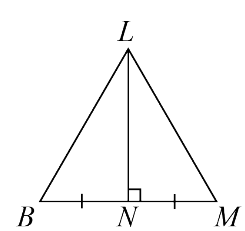
Label the endpoints of the line segment closest to B as L and M. Observe that $\triangle ABC$ can be tiled with small equilateral triangles congruent to $\triangle BLM$. That is, equilateral triangles with side length 18 mm. A complete justification of this is not provided here but you may wish to verify this for yourself.



In the second section from B there are 3 small equilateral triangles. In the third section there are 5 small equilateral triangles. In the fourth section there are 7 small equilateral triangles, and so on. The shaded region contains 20 small equilateral triangles. To find the area of the shaded region, we will find the area of the small equilateral triangle.

In $\triangle BLM$, drop a perpendicular from L to N on BM. Since $\triangle BLM$ is equilateral, N is the midpoint of BM and it follows that $BN = \frac{1}{2}BM = \frac{1}{2}(18) = 9$ mm.

Using the Pythagorean Theorem in $\triangle BLN$,



$$LN^2 = BL^2 - BN^2 = 18^2 - 9^2 = 243$$

Therefore $LN = \sqrt{243} = 9\sqrt{3}$ mm, since $LN \geq 0$.

Therefore, the area of $\triangle BLM$ is $\frac{(BM)(LN)}{2} = \frac{(18)(9\sqrt{3})}{2} = 81\sqrt{3}$ mm^2. Thus, the area of the blue trapezoid is $20 \times 81\sqrt{3} = 1620\sqrt{3}$ mm^2.

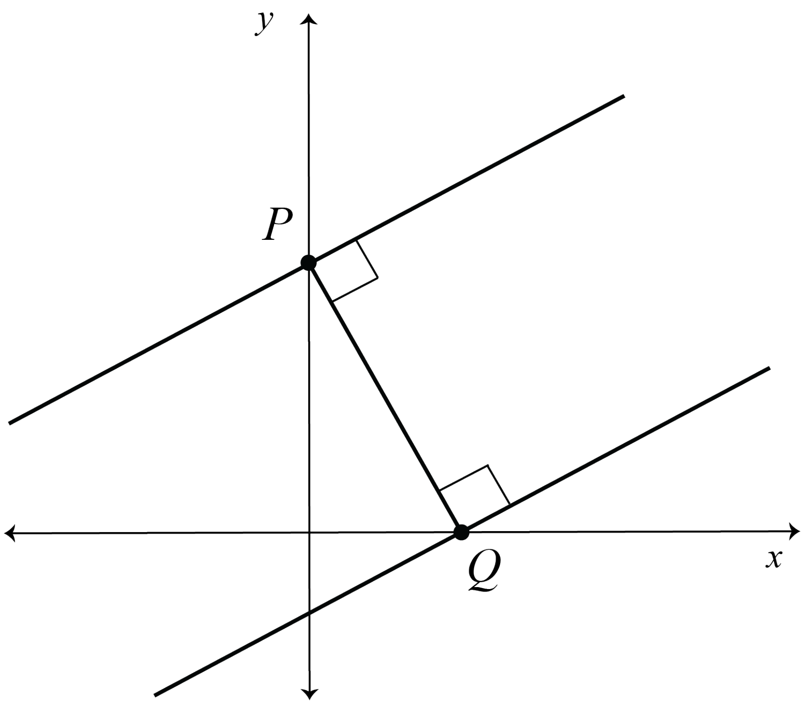
Problem of the Week

Problem D

Parallel Puzzle

Two distinct lines are drawn such that the first line passes through point P on the positive y-axis and the second line passes through point Q on the positive x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



Suggestion: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

Problem of the Week

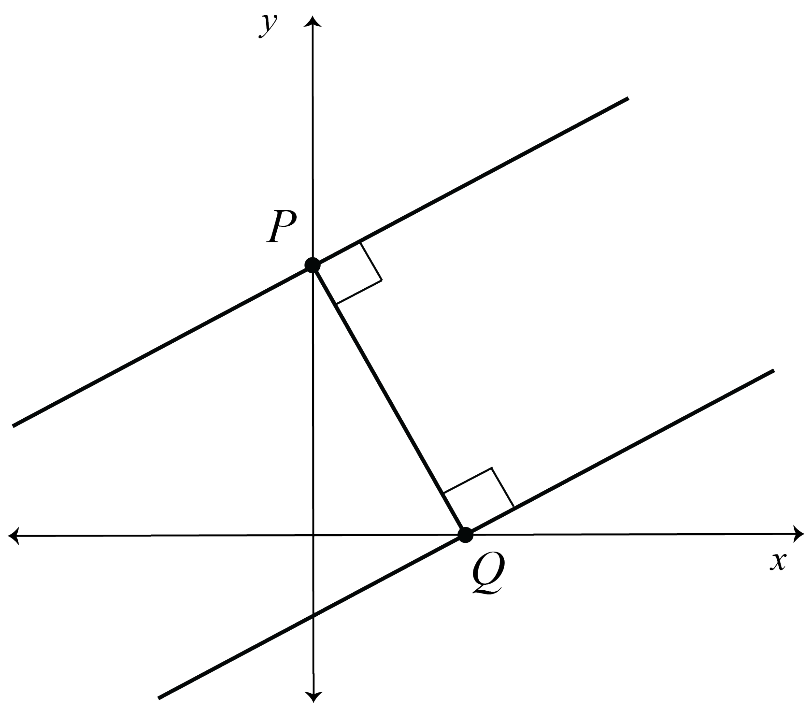
Problem D and Solution

Parallel Puzzle

Problem

Two distinct lines are drawn such that the first line passes through point P on the positive y-axis and the second line passes through point Q on the positive x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



SUGGESTION: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

Solution

We let l_1 represent the first line, which passes through P, and let l_2 represent the second line, which passes through Q.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P has coordinates $(0, k)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{m}$. Since k is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{m}x + k$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of the line through PQ is mk and Q has coordinates $(mk, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is m. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = mx + b$. Since $Q(mk, 0)$ lies on l_2, we have $0 = m(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2. A solution follows.

Let l_1 represent the line with equation $y = 4x + 3$. Let l_2 represent the second line, which passes through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P has coordinates $(0, 3)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{4}$. Since 3 is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{4}x + 3$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result $x = 12$ follows. Therefore, the x-intercept of the line through PQ is 12 and Q has coordinates $(12, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is 4. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = 4x + b$. Since $Q(12, 0)$ lies on l_2, we have $0 = 4(12) + b$ which simplifies to $b = -48$.

Therefore, the equation of l_2 is $y = 4x - 48$ and the y-intercept of l_2 is -48.

Number Sense (N)

Take me to the cover

Problem of the Week

Problem D

The Twos and Fives Dilemma

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks of bills that each have a total value of exactly \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

$2 Problem of the Week $5

Problem D and Solution

The Twos and Fives Dilemma

Problem

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks of bills that each have a total value of exactly \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

Solution

Consider a stack of bills with a total value of \$100 that includes x \$2 bills and y \$5 bills. The \$2 bills are worth $\$2x$ and the \$5 bills are worth $\$5y$, and so $2x + 5y = 100$.

Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$ or $5y = 100 - 2x$. (We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$2 bill and at least one \$5 bill.)

Since $x \geq 1$, then

$$\begin{aligned} 2x &\geq 2 \\ -2x &\leq -2 \\ 100 - 2x &\leq 100 - 2 \\ 100 - 2x &\leq 98 \end{aligned}$$

Also, since $5y = 100 - 2x$, this becomes $5y \leq 98$.

This means that $y \leq \frac{98}{5} = 19.6$. Since y is an integer, then $y \leq 19$.

Notice that since $5y = 100 - 2x$, then the right side is the difference between two even integers and is therefore even. This means that $5y$ (the left side) is even, which means that y must be even.

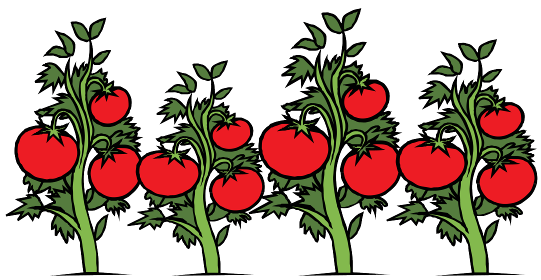
Since y is even, $y \geq 1$, and $y \leq 19$, then the possible values of y are 2, 4, 6, 8, 10, 12, 14, 16, and 18.

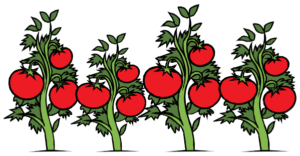
Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$. These ordered pairs are $(45, 2)$, $(40, 4)$, $(35, 6)$, $(30, 8)$, $(25, 10)$, $(20, 12)$, $(15, 14)$, $(10, 16)$, and $(5, 18)$.

Therefore, the maximum number of stacks that Virat can create is 9.

Problem of the Week
Problem D
Multiplying Tomatoes 1

Jamal grew four tomato plants last summer. The product of the total number of tomatoes each plant produced is 40 392. The plant that produced the most tomatoes produced exactly 50 more tomatoes than the plant that produced the fewest tomatoes. If the plant that produced the fewest tomatoes produced fewer than 10 tomatoes, determine all possibilities for the number of tomatoes each plant produced.





Problem of the Week
Problem D and Solution
Multiplying Tomatoes 1

Problem

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Solution

This problem is asking us to find four positive integers with a product of 40 392, where the difference between the smallest and largest integer is 50, and the smallest integer is less than 10. We can start by factoring 40 392. This gives $40\,392 = 1 \times 2^3 \times 3^3 \times 11 \times 17$. We have included 1 since it could be one of the positive integers.

Since the smallest integer is less than 10, from the factorization we can see that it could be 1, 2, 3, 4, 6, 8, or 9. We will consider these cases.

- If the smallest integer is 1, then the largest integer must be 51. Since $51 = 3 \times 17$, it follows that 51 is a factor of 40 392. The product of the other two integers must then be $2^3 \times 3^2 \times 11 = 792$.

  Thus, we are looking for two numbers, both less than 51, that multiply to 792. The factor pairs of 792 with both factors less than 51 are: 18 and 44, 22 and 36, and 24 and 33.

  Therefore, the four integers could be

  - 1, 18, 44, and 51
  - 1, 22, 36, and 51
  - 1, 24, 33, and 51

- If the smallest integer is 2, then the largest integer must be 52. However 52 is *not* a factor of 40 392, so this is not a possibility.

- If the smallest integer is 3, then the largest integer must be 53. However 53 is *not* a factor of 40 392, so this is not a possibility.

- If the smallest integer is 4, then the largest integer must be 54. Since $54 = 2 \times 3^3$, it follows that 54 is a factor of 40 392. The product of the other two integers must then be 11×17. Thus, the four integers could be 4, 11, 17, and 54.

- If the smallest integer is 6, then the largest integer must be 56. However 56 is *not* a factor of 40 392, so this is not a possibility.
- If the smallest integer is 8, then the largest integer must be 58. However 58 is *not* a factor of 40 392, so this is not a possibility.
- If the smallest integer is 9, then the largest integer must be 59. However 59 is *not* a factor of 40 392, so this is not a possibility.

Therefore, there are four possibilities for the number of tomatoes each plant produced. They are as follows:

- 1, 18, 44, and 51
- 1, 22, 36, and 51
- 1, 24, 33, and 51
- 4, 11, 17, and 54

Problem of the Week

Problem D

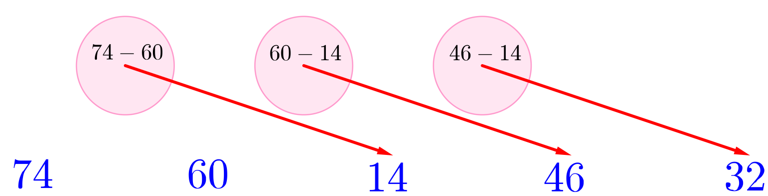
Difference Detective

The *non-negative difference* between two numbers a and b is $b - a$ or $a - b$, whichever is greater than or equal to 0. For example, the non-negative difference between 24 and 64 is 40.

Consider the sequence with first term 74, second term 60, and each term after equal to the non-negative difference between the previous two terms. That is, the third term is $74 - 60 = 14$, the fourth term is $60 - 14 = 46$, and the fifth term is $46 - 14 = 32$. So, the first five terms in the sequence are:

$$74, 60, 14, 46, 32$$

Determine the sum of the first 1306 terms in the sequence.



Problem of the Week
Problem D and Solution
Difference Detective

Problem

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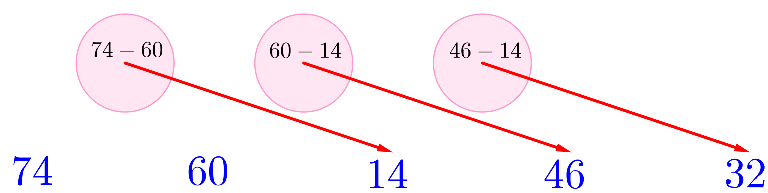

Solution

We start by generating more terms in the sequence in an attempt to find a pattern.

Using the rule for creating the sequence, we determine that the first 23 terms are

$$74,\ 60,\ 14,\ 46,\ 32,\ 14,\ 18,\ 4,\ 14,\ 10,\ 4,\ 6,\ 2,\ 4,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0,\ 2,\ 2,\ 0$$

The first 14 terms in the sequence have no apparent pattern. Afterwards, the terms repeat 2, 2, 0. Since each term is determined by the previous two, this sequence of 2, 2, 0 will repeat indefinitely.

Thus, the first 14 terms in the sequence are followed by n groups of 2, 2, 0. What is the value of n? We want a total of 1306 terms. If we remove the first 14 terms, we require $1306 - 14 = 1292$ more terms. If we divide 1292 by 3 we are able to determine how many complete sequences of 2, 2, 0 are needed. Since $1292 \div 3 = 430\frac{2}{3}$, there will be 430 complete sequences of 2, 2, 0 followed by 2, 2.

The required sum is the sum of the first 14 terms plus $430 \times (2 + 2 + 0) + 2 + 2$. The sum of the first 14 terms is 302. Thus, the sum of the first 1306 terms in the sequence is $302 + 430 \times 4 + 4 = 2026$.

Therefore, the sum of the first 1306 terms in the sequence is 2026.

Problem of the Week

Problem D

Math Test Math 2

Milana has had six tests in her math class so far. She knows the average (mean) of some of her marks.

- The average of her first and second test marks is 85%.
- The average of her second and third test marks is 82%.
- The average of her third and fourth test marks is 74%.
- The average of her fourth and fifth test marks is 79%.
- The average of her fifth and sixth test marks is 80%.

Determine the average of her first and sixth test marks.

Problem of the Week
Problem D and Solution
Math Test Math 2

Problem

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- The average of her first and second test marks is 85%.
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- The average of her third and fourth test marks is 74%.
- The average of her fourth and fifth test marks is 79%.
- The average of her fifth and sixth test marks is 80%.

Determine the average of her first and sixth test marks.

Solution

Let a, b, c, d, e, and f represent Milana's six test marks, in order. We are looking for the average of a and f. We know the following.

- The average of her first and second test marks is 85, so $\frac{a+b}{2} = 85$, which gives $a + b = 170$.
- The average of her second and third test marks is 82, so $\frac{b+c}{2} = 82$, which gives $b + c = 164$.
- The average of her third and fourth test marks is 74, so $\frac{c+d}{2} = 74$, which gives $c + d = 148$.
- The average of her fourth and fifth test marks is 79, so $\frac{d+e}{2} = 79$, which gives $d + e = 158$.
- The average of her fifth and sixth test marks is 80, so $\frac{e+f}{2} = 80$, which gives $e + f = 160$.

Adding these six equations gives the following:

$$\begin{aligned}
(a+b)+(b+c)+(c+d)+(d+e)+(e+f) &= 170+164+148+158+160 \\
a+2b+2c+2d+2e+f &= 800 \\
a+2(b+c)+2(d+e)+f &= 800 \\
a+2(164)+2(158)+f &= 800 \\
a+328+316+f &= 800 \\
a+f &= 156 \\
\frac{a+f}{2} &= 78
\end{aligned}$$

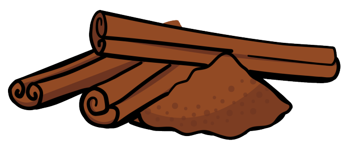
Therefore, the average of Milana's first and sixth test marks is 78%.

Problem of the Week

Problem D

A Sweet Mix

Mariana is buying cinnamon sugar, which is a mix made of cinnamon and sugar. There are two kinds available; one is in a red package and the other is in a blue package, but both packages have the same mass. In the red package, the ratio of cinnamon to sugar is $1 : 3$, by mass. In the blue package, the ratio of cinnamon to sugar is $1 : 4$, by mass. Mariana buys one red package and two blue packages, then mixes them together to create a new cinnamon sugar mix. Determine the ratio of cinnamon to sugar in Mariana's new cinnamon sugar mix.



Problem of the Week
Problem D and Solution
A Sweet Mix

Problem

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Solution

Solution 1

Let x represent the mass of each package of cinnamon sugar. Then the total mass of one red package of mix is x and the total mass of two blue packages of mix is $2x$.

Since the ratio of cinnamon to sugar in the red package, by mass, is $1 : 3$, then $\frac{1}{4}$ of the mass of the red package mix is cinnamon. It follows that the total mass of cinnamon in the red package mix is $\frac{1}{4}x$ and the total mass of sugar in the red package mix is $\frac{3}{4}x$.

Similarly, since the ratio of cinnamon to sugar in the blue package, by mass, is $1 : 4$, then $\frac{1}{5}$ of the mass of the blue package mix is cinnamon. It follows that the total mass of cinnamon in two blue packages of mix is $\frac{1}{5}(2x) = \frac{2}{5}x$ and the total mass of sugar in two blue packages of mix is $\frac{4}{5}(2x) = \frac{8}{5}x$.

When the packages are mixed to create the new cinnamon sugar mix, we obtain the following:

- Total mass of cinnamon: $\frac{1}{4}x + \frac{2}{5}x = \frac{5}{20}x + \frac{8}{20}x = \frac{13}{20}x$
- Total mass of sugar: $\frac{3}{4}x + \frac{8}{5}x = \frac{15}{20}x + \frac{32}{20}x = \frac{47}{20}x$

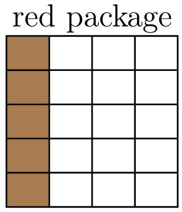
Thus, the ratio of cinnamon to sugar is $\frac{13}{20}x : \frac{47}{20}x = 13 : 47$.

Solution 2

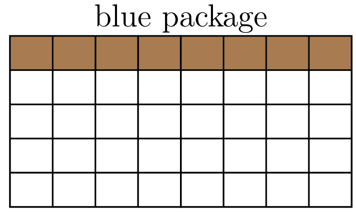
Let $\frac{1}{20}$ of each package be a unit of mass. Since the ratio of cinnamon to sugar in the red package, by mass, is $1 : 3$, then $\frac{1}{4}$ of the package, or 5 units of mass are cinnamon. Then the remaining 15 units are sugar.

Since there are two blue packages, then there are 40 units of mass in total of the blue package mix. Since the ratio of cinnamon to sugar in the blue package, by mass, is $1 : 4$, then $\frac{1}{5}$ of the mix, or 8 units of mass are cinnamon. Then the remaining 32 units are sugar. This is illustrated in the following diagrams, where the units of cinnamon are shaded.

red package



blue package



When the packages are combined, there is a total of 60 units of mass. Of these, $5 + 8 = 13$ are cinnamon and $15 + 32 = 47$ are sugar. Thus, the ratio of cinnamon to sugar is $13 : 47$.

Problem of the Week

Problem D

Come Get Your Tickets!

POTW Secondary School is putting on a play. Tickets for the play are numbered, and each ticket consists of exactly four digits chosen from the digits 0 to 9 (digits may be repeated).

Every possible ticket is printed exactly once. If tickets are handed out in a random order, what is the probability that first ticket handed out has digits whose sum is 34 or higher?

Problem of the Week

Problem D and Solution

Come Get Your Tickets!

Problem

POTW Secondary School is putting on a play. Tickets for the play are numbered, and each ticket consists of exactly four digits chosen from the digits 0 to 9 (digits may be repeated).

Every possible ticket is printed exactly once. If tickets are handed out in a random order, what is the probability that first ticket handed out has digits whose sum is 34 or higher?

Solution

First, we determine the total number of possible four-digit ticket numbers. There are 10 choices for the first digit. For each of these possibilities, there are 10 choices for the second digit. Therefore, there are $10 \times 10 = 100$ possibilities for the first two digits. For each of these possibilities, there are 10 choices for the third digit. Therefore, there are $100 \times 10 = 1000$ possibilities for the first three digits. And finally, for each of these 1000 possibilities for the first three digits, there are 10 choices for the fourth digit. Therefore, there are $1000 \times 10 = 10\,000$ possible four-digit ticket numbers.

Now, we determine the number of ticket numbers with a digit sum of 34 or higher. We consider cases.

- **Case 1:** The ticket number contains four 9s.
  If the digits are all 9s, the sum is 36, which is 34 or higher. There is one ticket with all 9s for digits.
- **Case 2:** The ticket number contains three 9s and one other digit.
  The three 9s sum to 27. In order to get a sum of 34 or higher, the fourth digit must be 7 or 8. Thus, there are two choices for the fourth digit. Once the digit is chosen, there are four places to put the digit. Once the digit is placed, the digits in the remaining spots must be 9s. Therefore, there are $2 \times 4 = 8$ ticket numbers containing three 9s. (It is possible to list them: 7999, 8999, 9799, 9899, 9979, 9989, 9997, 9998.)

- **Case 3:** The ticket number contains two 9s.
  The two 9s sum to 18. In order to get a sum of 34 or higher, we need a sum of $34 - 18 = 16$ or higher from the two remaining digits. The only way to do this, since we cannot use more 9s, is to use two 8s. The two 8s can be placed in six ways and then the 9s must go in the remaining spots. Therefore, there are six ticket numbers containing two 9s. (It is possible to list them: 8899, 8989, 8998, 9889, 9898, 9988.)

- **Case 4:** The ticket number contains one 9.
  In order to get a sum of 34 or higher, we need a sum of $34 - 9 = 25$ or higher from the remaining three digits. But this sum would be made from three digits chosen from the digits 0 to 8. The maximum possible sum would be 24 if three 8s were used. We need the sum to be 25 or higher. Therefore, no ticket number containing only one 9 will produce a sum of 34 or higher.
  It should be noted that a ticket number with zero nines would not produce a sum of 34 or higher either.

Therefore, the number of ticket numbers with a digit sum of 34 or higher is $1 + 8 + 6 = 15$. To calculate the probability we divide the number of tickets with a digit sum of 34 or higher by the number of possible ticket numbers. Therefore, the probability that the first ticket handed out has digits whose sum is 34 or higher is $\frac{15}{10\,000} = \frac{3}{2000}$.

Another way of looking at this result is that out of every 2000 tickets you could expect, on average, 3 tickets with a digit sum of 34 or higher.

Problem of the Week

Problem D

Squares in a Square

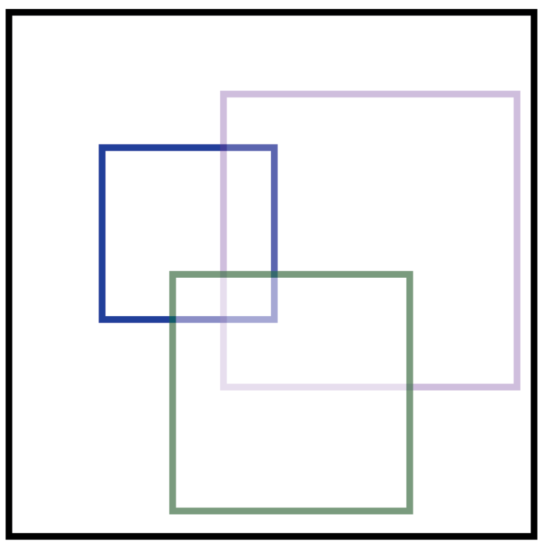
The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 36^3 are perfect squares?



Problem of the Week
Problem D and Solution
Squares in a Square

Problem

The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 36^3 are perfect squares?

Solution

First, let's look at the prime factorization of four different perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \ 129\,600 = 2^6 \times 3^4 \times 5^2$$

Note that, in each case, the exponent on each of the prime factors is even. In fact, a positive integer is a perfect square exactly when the exponent on each of the prime factors in its prime factorization is an even integer greater than or equal to zero. Now

$$\begin{aligned} 36^3 &= (2^2 \times 3^2)^3 \\ &= (2^2)^3 \times (3^2)^3 \\ &= 2^6 \times 3^6 \end{aligned}$$

All positive divisors of 36^3 will be of the form $2^k \times 3^n$ where k and n are integers with $0 \leq k \leq 6$ and $0 \leq n \leq 6$.

For $2^k \times 3^n$ to be a perfect square, k and n must be even integers. Thus, $k \in \{0, 2, 4, 6\}$ and $n \in \{0, 2, 4, 6\}$.

For each of the 4 values of k, there are 4 values of n, so there are $4 \times 4 = 16$ perfect square divisors of 36^3.

Therefore, 36^3 has 16 divisors that are perfect squares.

We can systematically list all of the divisors that are perfect squares. They are:

$2^0 3^0 = 1$	$2^0 3^2 = 9$	$2^0 3^4 = 81$	$2^0 3^6 = 729$
$2^2 3^0 = 4$	$2^2 3^2 = 36$	$2^2 3^4 = 324$	$2^2 3^6 = 2916$
$2^4 3^0 = 16$	$2^4 3^2 = 144$	$2^4 3^4 = 1296$	$2^4 3^6 = 11\,664$
$2^6 3^0 = 64$	$2^6 3^2 = 576$	$2^6 3^4 = 5184$	$2^6 3^6 = 46\,656$

Problem of the Week
Problem D
Number Triangle

Consecutive positive integers are arranged in rows into the shape of a triangle. In this triangle, the top row contains the integer 1, and each row below the top row contains one more integer than the row above, starts with the next consecutive integer after the largest integer in the row above, and lists the consecutive integers in increasing order. That is, the first row contains the integer 1, the second row contains the integers 2 and 3, the third row contains the integers 4, 5, and 6, the fourth row contains the integers 7, 8, 9, and 10, and so on.

In which row will the integer 2026 appear? What integers are in this row?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 3 & & & \\ & & 4 & & 5 & & 6 & & \\ & 7 & & 8 & & 9 & & 10 & \\ 11 & & 12 & & 13 & & 14 & & 15 \\ & & & & \vdots & & & & \end{array}$$

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Problem of the Week

Problem D and Solution

Number Triangle

Problem

Consecutive positive integers are arranged in rows into the shape of a triangle. In this triangle, the top row contains the integer 1, and each row below the top row contains one more integer than the row above, starts with the next consecutive integer after the largest integer in the row above, and lists the consecutive integers in increasing order. That is, the first row contains the integer 1, the second row contains the integers 2 and 3, the third row contains the integers 4, 5, and 6, the fourth row contains the integers 7, 8, 9, and 10, and so on.

In which row will the integer 2026 appear? What integers are in this row?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & & 3 & & & \\ & & 4 & & 5 & & 6 & & \\ & 7 & & 8 & & 9 & & 10 & \\ 11 & & 12 & & 13 & & 14 & & 15 \\ & & & & \vdots & & & & \end{array}$$

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Solution

After the first row, there is 1 integer in the triangle.

After the second row, there are $1+2=3$ integers in the triangle.

After the third row, there are $1+2+3=6$ integers in the triangle.

After the fourth row, there are $1+2+3+4=10$ integers in the triangle.

After the fifth row, there are $1+2+3+4+5=15$ integers in the triangle.

After the n^{th} row, there are $1+2+3+\cdots+n$ integers in the triangle.

If 2026 is in row n, then $1+2+3+\cdots+n \geq 2026$ and $1+2+3+\cdots+(n-1) < 2026$.

To determine n, let's try solving $1+2+\cdots+n=2026$.

It is given that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Thus, we are looking to solve $\frac{n(n+1)}{2} = 2026$, or equivalently $n^2 + n = 4052$.

At this point we will use trial and error. At the end of the solution, a more algebraic approach using the quadratic formula is presented.

Suppose $n = 25$. Then $n^2 + n = 25^2 + 25 = 650 < 4052$.

We can try a larger value for n. Suppose $n = 50$. Then, $n^2 + n = 50^2 + 50 = 2550 < 4052$.

We can try another yet larger value for n. Suppose $n = 65$. Then, $n^2 + n = 65^2 + 65 = 4290 > 4052$.

We are now close. We can try a smaller value for n. Suppose $n = 63$. Then, $n^2 + n = 63^2 + 63 = 4032 < 4052$.

Since this value is very close to 4052, we can then try the next integer, so let $n = 64$. Then, $n^2 + n = 64^2 + 64 = 4160 > 4052$.

This tells us that the integer 2026 is in the 64^{th} row. Further, the first 63 rows contain the integers from 1 to $\frac{63(64)}{2} = 2016$. Therefore, the 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.

We will finish by showing how we can find the value of n algebraically. We will first find the value of $n, n > 0$, so that

$$\begin{aligned} \frac{n(n+1)}{2} &= 2026 \\ n(n+1) &= 4052 \\ n^2 + n - 4052 &= 0 \end{aligned}$$

The quadratic formula can be used to solve for n.

$$\begin{aligned} n &= \frac{-1 \pm \sqrt{1 - 4(1)(-4052)}}{2} \\ n &= \frac{-1 \pm \sqrt{16\,209}}{2} \end{aligned}$$

It follows that $n \approx 63.157$ or $n \approx -64.157$. Since $n \approx -64.157 < 0$, it is inadmissible. Then $n \approx 63.157$.

Notice that $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2026$, and $1 + 2 + 3 + \cdots + 64 = \frac{64(65)}{2} = 2080 \geq 2026$.

Thus, the integer 2026 is in the 64^{th} row. The 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.

Problem of the Week
Problem D
The Flavour Foundry

Salt and Pepper are chefs scaling up a recipe for a large banquet. The total amount of dried garlic needed is 60% more than what Salt has in stock and 80% more than what Pepper has in stock. After combining the amounts they have in stock, they find they have more than necessary. What percentage over the required amount do they have? Round your answer to the nearest tenth of a percent.

Problem of the Week

Problem D and Solution

The Flavour Foundry

Problem

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Solution

Let g be the total required amount needed in kg, s be Salt's amount, in kg, p be Pepper's amount, in kg.

Since the required amount is 60% more than the amount Salt has, then $g = 1.60s = \frac{8}{5}s$, or $s = \frac{5}{8}g$.

Similarly, since the required amount is 80% more than the amount Pepper has, $g = 1.80p = \frac{9}{5}p$, or $p = \frac{5}{9}g$.

Let t be the total amount Salt and Pepper have, in kg.

Now,

$$\begin{aligned} t &= s + p \\ &= \frac{5}{8}g + \frac{5}{9}g \\ &= \left(\frac{5}{8} + \frac{5}{9}\right) g \\ &= \left(\frac{45}{72} + \frac{40}{72}\right) g \\ &= \frac{85}{72}g \\ &= \left(1 + \frac{13}{72}\right) g \end{aligned}$$

Therefore, the total amount Salt and Pepper have exceeds the required amount by $\frac{13}{72} \times 100\%$, which is approximately 18.1%.

Problem of the Week

Problem D

Just Like Kayak 2

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine the number of six-digit palindromic integers that are divisible by 15.

NOTE: An integer is divisible by 3 exactly when the sum of its digits is divisible by 3. For example, 15 972 is divisible by 3 since $1 + 5 + 9 + 7 + 2 = 24$ and 24 is divisible by 3. But 14 923 is not divisible by 3 since $1 + 4 + 9 + 2 + 3 = 19$ and 19 is not divisible by 3.

Problem of the Week
Problem D and Solution
Just Like Kayak 2

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Solution

We are looking for six-digit integers of the form $abccba$ that are divisible by 15. For an integer to be divisible by 15, it must be divisible by both 3 and 5.

To be divisible by 5, an integer must end in 0 or 5. If a palindrome ends in 0, it must also begin with 0. However, the integer $0bccb0$ is not a six-digit integer since the leading digit is 0. Therefore, the palindromes cannot end with a 0 and hence must start and end with a 5. It follows that any six-digit palindrome divisible by 5 must be of the form $5bccb5$.

For an integer to be divisible by 3, the sum of its digits must be divisible by 3. Therefore, $5 + b + c + c + b + 5 = 10 + 2b + 2c$ must be divisible by 3. Since b and c are digits, we note that $0 \leq b, c \leq 9$, where b and c are integers. This leads to $10 \leq 10 + 2b + 2c \leq 46$.

The integers between 10 and 46 that are divisible by 3 are 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, and 45. If we choose 12 to be the sum of the digits, then

$$\begin{aligned} 10 + 2b + 2c &= 12 \\ 2(b + c) &= 2 \\ b + c &= 1 \end{aligned}$$

This equation has solutions $b = 1,\ c = 0$ or $b = 0,\ c = 1$.

If we choose 15 to be the sum of the digits, then

$$\begin{aligned} 10 + 2b + 2c &= 15 \\ 2b + 2c &= 5 \\ 2(b + c) &= 2.5 \end{aligned}$$

This equation has no solution, since b and c are integers. Similarly, there will be no solutions for any of the odd sums.

Therefore, we only need to solve the equations with even sums. The results are summarized in the following table.

Original Equation	**Simplified Equation**	**Solutions in the form** (b, c)	**Number of Solutions**
$10 + 2b + 2c = 12$	$b + c = 1$	$(0, 1)$, $(1, 0)$	2
$10 + 2b + 2c = 18$	$b + c = 4$	$(0, 4)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, $(4, 0)$	5
$10 + 2b + 2c = 24$	$b + c = 7$	$(0, 7)$, $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, $(6, 1)$, $(7, 0)$	8
$10 + 2b + 2c = 30$	$b + c = 10$	$(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$	9
$10 + 2b + 2c = 36$	$b + c = 13$	$(4, 9)$, $(5, 8)$, $(6, 7)$, $(7, 6)$, $(8, 5)$, $(9, 4)$	6
$10 + 2b + 2c = 42$	$b + c = 16$	$(7, 9)$, $(8, 8)$, $(9, 7)$	3

Therefore, the total number of six-digit palindromic integers divisible by 15 is $2 + 5 + 8 + 9 + 6 + 3 = 33$.

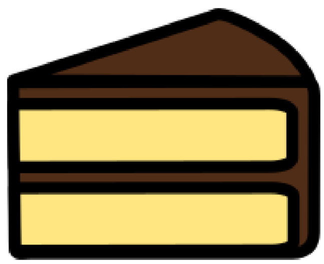
Problem of the Week

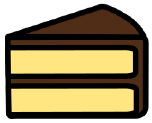
Problem D

Cheesecake Geometry

For Amanda's birthday, Rhett made a chocolate cheesecake. The cake was in the shape of a cylinder, with equal radius and height. Rhett cut the cake into 8 congruent slices, each in the shape of a sector of a cylinder. Rhett then ate one slice for quality control purposes.

After removing Rhett's slice, by what percentage has the surface area of the cake increased or decreased? Round your answer to one decimal place.





Problem of the Week
Problem D and Solution
Cheesecake Geometry

Problem

For Amanda's birthday, Rhett made a chocolate cheesecake. The cake was in the shape of a cylinder, with equal radius and height. Rhett cut the cake into 8 congruent slices, each in the shape of a sector of a cylinder. Rhett then ate one slice for quality control purposes.

After removing Rhett's slice, by what percentage has the surface area of the cake increased or decreased? Round your answer to one decimal place.

Solution

Let the cake have radius r and height h. We are given that $r = h$.

We first calculate the surface area of the entire cake without the slice removed.

The surface area of a cylinder includes the areas of the two circular faces plus the surface area of the side. The circular faces each have area equal to πr^2. Since the circumference of a circular face is $2\pi r$, the surface area of the side is $(2\pi r)(h) = 2\pi r^2$, since $r = h$. Thus, the total surface area of the cake before the slice is removed is $\pi r^2 + \pi r^2 + 2\pi r^2 = 4\pi r^2$.

We now calculate the surface area of the cake after the slice is removed.

The surface area removed includes $\frac{1}{8}$ of the area of each of the two circular faces with radius r and $\frac{1}{8}$ of the surface area of the side of the cylinder. Therefore, the surface area remaining includes $1 - \frac{1}{8} = \frac{7}{8}$ of the original surface area. That is, $\frac{7}{8} \times 4\pi r^2 = \frac{7\pi r^2}{2}$.

There are two areas that are added as a result of removing the slice. These areas are the rectangular faces of the slice that were inside the cylindrical cake before the slice was removed. These rectangles each have length equal to the radius of the circular face, r, and width $h = r$. Thus, the surface area added is equal to $2(r)(r) = 2r^2$.

Therefore, the new total surface area is equal to $\frac{7\pi r^2}{2} + 2r^2$.

The change in surface area is $(\frac{7\pi r^2}{2} + 2r^2) - 4\pi r^2 = -\frac{\pi r^2}{2} + 2r^2 = r^2(2 - \frac{\pi}{2})$.

Now, $\frac{\pi}{2} < 2$, so $2 - \frac{\pi}{2} > 0$. Therefore, the surface area increases as a result of removing the slice.

To calculate the percentage that the area has increased, we divide the surface area increase by the original surface area, and multiply by 100%.

The increase is $r^2\left(2 - \frac{\pi}{2}\right)$. Therefore, the percentage increase in surface area is equal to

$$\begin{aligned}\frac{r^2\left(2 - \frac{\pi}{2}\right)}{4\pi r^2} \times 100\% &= \frac{\left(2 - \frac{\pi}{2}\right)}{4\pi} \times 100\% \\ &= \left(\frac{4 - \pi}{8\pi}\right) \times 100\% \\ &\approx 3.4\%\end{aligned}$$

Therefore, the surface area of the cake increases by approximately 3.4% after the slice has been removed.

Problem of the Week

Problem D

Favourite Number 2

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Ayumi announced that her favourite number is a three-digit positive integer with a digit sum of 9. Vinnie then wrote down a three-digit positive integer with a digit sum of 9. What is the probability that the number Vinnie wrote was Ayumi's favourite number?

Problem of the Week
Problem D and Solution
Favourite Number 2

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Ayumi announced that her favourite number is a three-digit positive integer with a digit sum of 9. Vinnie then wrote down a three-digit positive integer with a digit sum of 9. What is the probability that the number Vinnie wrote was Ayumi's favourite number?

Solution

There are twelve triples of digits that sum to 9. These are: $(9, 0, 0)$, $(8, 1, 0)$, $(7, 2, 0)$, $(7, 1, 1)$, $(6, 3, 0)$, $(6, 2, 1)$, $(5, 4, 0)$, $(5, 3, 1)$, $(5, 2, 2)$, $(4, 4, 1)$, $(4, 3, 2)$, and $(3, 3, 3)$.

We will count the number of three-digit positive integers whose digit sum is 9 by counting the number of positive integers formed by the digits in each triple above. We will consider cases based on the number of zeroes in each triple, since a three-digit integer cannot have a leading digit of 0.

- **Case 1:** Two zeroes
  The only triple with two zeros is $(9, 0, 0)$. Since the first digit of the three-digit integer cannot be zero, the only possible three-digit integer is 900. Thus, there is only 1 possible three-digit integer with two zeroes.

- **Case 2:** One zero
  The triples with one zero are $(8, 1, 0)$, $(7, 2, 0)$, $(6, 3, 0)$, and $(5, 4, 0)$. Consider $(8, 1, 0)$. Since the zero cannot be in the first position, there are four integers that can be formed with these three digits: 810, 801, 180, and 108. Since this is true for any of the four triples in this case, there are $4 \times 4 = 16$ possible three-digit integers with one zero.

- **Case 3:** No zeroes
  In this case, we will further separate the triples based on the number of repeated digits.

  - **Case 3a:** Three repeated digits
    The only triple with three repeated digits is $(3, 3, 3)$ and the only integer formed by these digits is 333. Thus, there is only 1 possible three-digit integer in this case.

- **Case 3b:** Two repeated digits
  The triples with two repeated digits are $(7, 1, 1)$, $(5, 2, 2)$, and $(4, 4, 1)$. Consider $(7, 1, 1)$. There are three integers that can be formed by these three digits: 711, 171, and 117. Since this is true for any of the three triples in this case, there are $3 \times 3 = 9$ possible three-digit integers in this case.
- **Case 3c:** No repeated digits
  The triples are $(6, 2, 1)$, $(5, 3, 1)$, and $(4, 3, 2)$. Consider $(6, 2, 1)$. There are six integers that can be formed by these three digits: 621, 612, 261, 216, 162, and 126. Since this is true for any of the three triples in this case, there are $3 \times 6 = 18$ possible three-digit integers in this case.

Therefore, there are $1 + 9 + 18 = 28$ possible three-digit integers with no zeroes.

Therefore, the total number of three-digit positive integers that have a digit sum of 9 is $1 + 16 + 28 = 45$. Since Vinnie wrote down only one of these integers, the probability that this integer was Ayumi's favourite number is $\frac{1}{45}$.

Note: It is a known fact that an integer is divisible by 9 exactly when its digit sum is divisible by 9. For example, 32 814 has a digit sum of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 9, then 32 814 is divisible by 9. On the other hand, 32 810 has a digit sum of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 9, then 32 810 is not divisible by 9.

As a consequence of this fact, Ayumi's favourite number must be divisible by 9.

Problem of the Week
Problem D
Fair Split Collective

A team of employees completed a large project. They were recognized with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

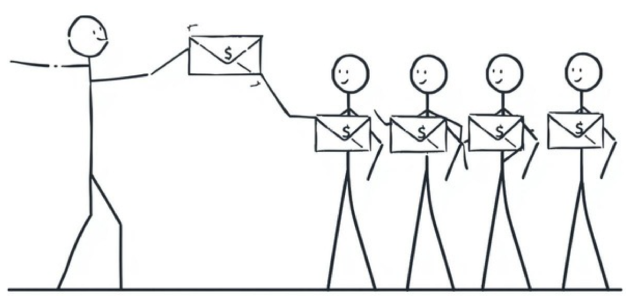
The bonus was paid out as follows:

(i) \$1000 to the oldest member of the project team plus $\frac{1}{10}$ of what remains, then

(ii) \$2000 to the second oldest member of the project team plus $\frac{1}{10}$ of what then remains, then

(iii) \$3000 to the third oldest member of the project team plus $\frac{1}{10}$ of what then remains, and so on.

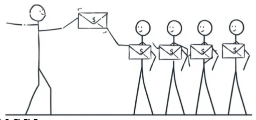
After all of the bonus money had been distributed, each member of the project team had received the same amount and no money was left over. What is the age of the youngest member of the project team?



Problem of the Week

Problem D and Solution

Fair Split Collective



Problem

A team of employees completed a large project. They were recognized with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

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After all of the bonus money had been distributed, each member of the project team had received the same amount and no money was left over. What is the age of the youngest member of the project team?

Solution

Solution 1

Let x represent the amount that each team member receives.
Let y represent the total amount of the bonus.
Then $y \div x$ is the number of team members.

The first team member gets \$1000 plus one-tenth of the remainder. Therefore,

$$x = 1000 + \frac{1}{10}(y - 1000)$$

Multiplying both sides by 10,

$$10x = 10\,000 + y - 1000$$

Simplifying and solving for y, we have

$$y = 10x - 9000$$

The second team member gets \$2000 plus one-tenth of the remainder after the first team member's share and \$2000 is removed. Therefore,

$$x = 2000 + \frac{1}{10}(y - x - 2000)$$

Multiplying both sides by 10,

$$10x = 20\,000 + y - x - 2000$$

Simplifying and solving for y, we have

$$y = 11x - 18\,000$$

Since $y = 10x - 9000$ and $y = 11x - 18\,000$, we have

$$\begin{aligned} 10x - 9000 &= 11x - 18\,000 \\ x &= 9000 \end{aligned}$$

Therefore, $y = 10x - 9000 = 10(9000) - 9000 = 81\,000$.

Therefore, each team member receives \$9000 and the total bonus is \$81 000. The number of team members is $y \div x = 81\,000 \div 9000 = 9$. The oldest team member is 45 and the ages of the team members are consecutive integers, so the youngest team member is 37. The ages of the team members are 37, 38, 39, 40, 41, 42, 43, 44, 45.

Solution 2

Let n represent the number of team members.

Team member n (the youngest team member) receives $n \times \$1000$ or $1000n$.

So $1000n$ is $\frac{9}{10}$ of what remains after team member $(n-1)$ is given $(n-1) \times 1000$.

Therefore, $\frac{1}{10}$ of what remains is $\frac{1000n}{9}$.

Then team member $(n-1)$ receives $1000(n-1) + \frac{1000n}{9}$.

But team member $(n-1)$ and team member n each receive the same amount.

Therefore,

$$\begin{aligned} 1000(n-1) + \frac{1000n}{9} &= 1000n \\ 1000n - 1000 + \frac{1000n}{9} &= 1000n \\ \frac{1000n}{9} &= 1000 \\ 1000n &= 9000 \\ n &= 9 \end{aligned}$$

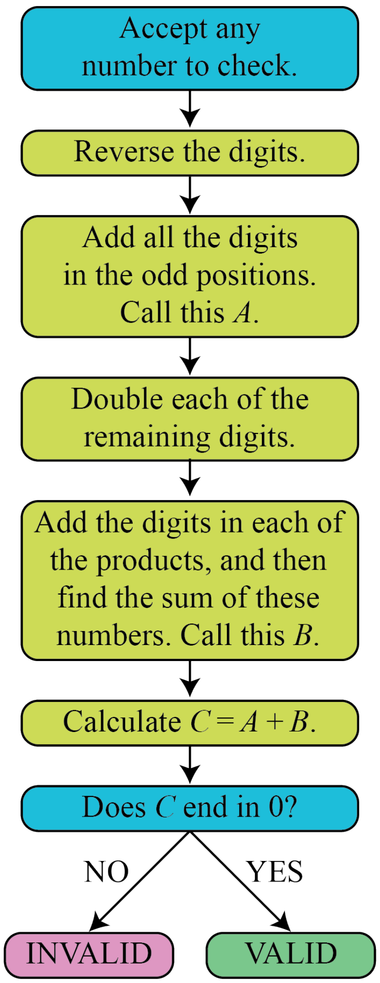
Therefore, there are 9 team members and each receives $\$1000n = \$1000(9) = \$9000$. Since the oldest team member is 45 and the ages of the team members are consecutive integers, the youngest team member is 37.

Problem of the Week

Problem D

Check Please 2

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$
  $$\begin{aligned} A &= 2+7+3 \\ &= 12 \end{aligned}$$
- Double remaining digits:
  $$2 \times 9 = 18$$
  $$2 \times 5 = 10$$
  $$2 \times 1 = 2$$
- Calculate B:
  $$\begin{aligned} B &= (1+8)+(1+0)+2 \\ &= 9+1+2 \\ &= 12 \end{aligned}$$
- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$
  $$\begin{aligned} A &= 7+9+5+1 \\ &= 22 \end{aligned}$$
- Double remaining digits:
  $$2 \times 8 = 16$$
  $$2 \times 7 = 14$$
  $$2 \times 3 = 6$$
- Calculate B:
  $$\begin{aligned} B &= (1+6)+(1+4)+6 \\ &= 7+5+6 \\ &= 18 \end{aligned}$$
- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 8764 $X6X5$ $X6X8$ 5409 is a valid card number when verified by the Luhn Algorithm, where X is a single digit. Determine all possible values of X.

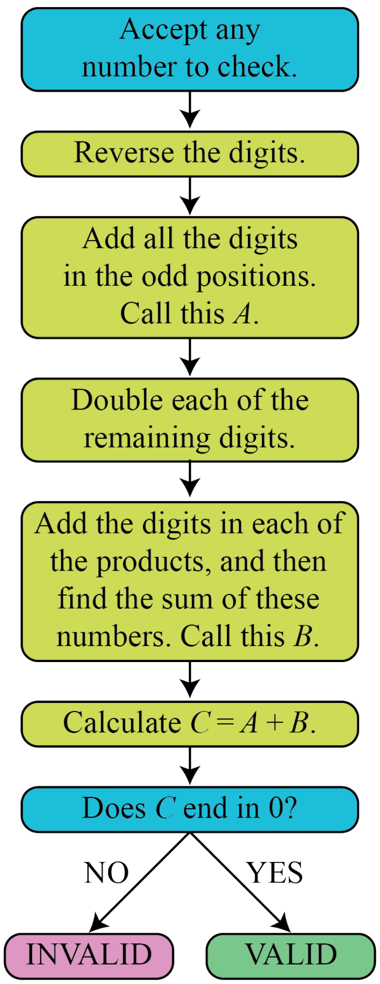
Problem of the Week

Problem D and Solution

Check Please 2

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
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- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$

- Calculate B:

$$\begin{aligned} B &= (1 + 8) + (1 + 0) + 2 \\ &= 9 + 1 + 2 \\ &= 12 \end{aligned}$$

- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number 8764 $X6X5$ $X6X8$ 5409 is a valid card number when verified by the Luhn Algorithm, where X is a single digit. Determine all possible values of X.

Solution

Solution 1

When the digits of the number are reversed the resulting number is 9045 $8X6X$ $5X6X$ 4678. The sum of the digits in the odd positions is

$$A = 9 + 4 + 8 + 6 + 5 + 6 + 4 + 7 = 49.$$

When the digits in the remaining positions are doubled, the following products are obtained:

$$2(0) = 0;\ 2(5) = 10;\ 2(X) = 2X;\ 2(X) = 2X;$$

$$2(X) = 2X;\ 2(X) = 2X;\ 2(6) = 12;\ 2(8) = 16$$

Let n represent the sum of the digits of $2X$.

When the digit sums from each of the products are added, the sum is:

$$B = 0 + (1+0) + n + n + n + n + (1+2) + (1+6) = 0 + 1 + 4n + 3 + 7 = 4n + 11.$$

Since $C = A + B$, we have $C = 49 + 4n + 11 = 60 + 4n$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what possible sums can be obtained? We summarize these in the following table.

Original Digit (X)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2X$)	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2X$ (n)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of $2X$ can be any integer from 0 to 9 inclusive. It follows that the only values for n are the integers from 0 to 9.

To be a valid number, the units digit of C must be 0. We want $60 + 4n$ to be an integer whose units digit is 0. Since the minimum value of n is 0, it follows that $60 + 4n \geq 60$. We will now look at cases based on the value of $60 + 4n$.

- Case 1: $60 + 4n = 60$.
  Then $4n = 0$, so $n = 0$. From our table, this occurs when $X = 0$. This is one possible value for X.

- Case 2: $60 + 4n = 70$.
  Then $4n = 10$, so $n = 2.5$. However n must be an integer value so this is not possible.

- Case 3: $60 + 4n = 80$.
  Then $4n = 20$, so $n = 5$. From our table, this occurs when $X = 7$. This is one possible value for X.
- Case 4: $60 + 4n = 90$.
  Then $4n = 30$, so $n = 7.5$. However n must be an integer value so this is not possible.
- Case 5: $60 + 4n = 100$.
  Then $4n = 40$, so $n = 10$. However n must be an integer from 0 to 9 inclusive, so this is not possible. This tells us that $60 + 4n < 100$, so there are no further cases.

Therefore, the two valid possibilities for X are 0 and 7.

When $X = 0$, the number is 8764 0605 0608 5409, which is indeed valid by the Luhn Algorithm.
When $X = 7$, the number is 8764 7675 7678 5409 which is indeed valid by the Luhn Algorithm.

Solution 2

The second solution looks at each of the possible values of X and then verifies the resulting number. A computer program or spreadsheet could be developed to solve this problem efficiently.

Remember that A is the sum of the digits in the odd positions of the reversal. Each of the digits in the even positions of the reversal are doubled, and B is the sum of the sum of the digits of each of these products. C is the sum $A + B$.

X	Number	Reversal	A	Double Even Digits	B	C	Valid?
0	8764 0605 0608 5409	9045 8060 5060 4678	49	0, 10, 0, 0, 0, 0, 12, 16	11	60	Yes
1	8764 1615 1618 5409	9045 8161 5161 4678	49	0, 10, 2, 2, 2, 2, 12, 16	19	68	No
2	8764 2625 2628 5409	9045 8262 5262 4678	49	0, 10, 4, 4, 4, 4, 12, 16	27	76	No
3	8764 3635 3638 5409	9045 8363 5363 4678	49	0, 10, 6, 6, 6, 6, 12, 16	35	84	No
4	8764 4645 4648 5409	9045 8464 5464 4678	49	0, 10, 8, 8, 8, 8, 12, 16	43	92	No
5	8764 5655 5658 5409	9045 8565 5565 4678	49	0, 10, 10, 10, 10, 10, 12, 16	15	64	No
6	8764 6665 6668 5409	9045 8666 5666 4678	49	0, 10, 12, 12, 12, 12, 12, 16	23	72	No
7	8764 7675 7678 5409	9045 8767 5767 4678	49	0, 10, 14, 14, 14, 14, 12, 16	31	80	Yes
8	8764 8685 8688 5409	9045 8868 5868 4678	49	0, 10, 16, 16, 16, 16, 12, 16	39	88	No
9	8764 9695 9698 5409	9045 8969 5969 4678	49	0, 10, 18, 18, 18, 18, 12, 16	47	96	No

Therefore, the two valid possibilities for X are 0 and 7.

Problem of the Week

Problem D

Sharing Beads

Golriz and Diane each have some beads. If Diane gives six of her beads to Golriz, then Golriz would then have twice as many beads as Diane. Instead, if Diane takes six beads from Golriz, then both would have the same number of the beads. What is the total number of beads that Golriz and Diane have?

Problem of the Week
Problem D and Solution
Sharing Beads

Problem

Golriz and Diane each have some beads. If Diane gives six of her beads to Golriz, then Golriz would then have twice as many beads as Diane. Instead, if Diane takes six beads from Golriz, then both would have the same number of the beads. What is the total number of beads that Golriz and Diane have?

Solution

Solution 1

Let d be the number of beads that Diane has, and let g be the number of beads that Golriz has.

If Diane gives 6 beads to Golriz, then Diane would have $d-6$ beads and Golriz would have $g+6$ beads. Then Golriz has twice as many beads as Diane, meaning $g+6=2(d-6)$. Thus, $g=2d-18$.

If Diane takes 6 beads from Golriz, then Diane would have $d+6$ beads and Golriz would have $g-6$ beads. Then they both would have the same number of beads, meaning $d+6=g-6$. Thus, $g=d+12$.

Since $g=2d-18$ and $g=d+12$, we have $2d-18=d+12$ or $d=30$. Thus, $g=d+12=30+12=42$.

Thus, the total number of beads that Golriz and Diane have is $30+42=72$.

Solution 2

Let d be the number of beads that Diane has.

If Diane takes 6 beads from Golriz, then the two of them would have the same number of beads. This tells us that Golriz has 12 more beads than Diane, or that Golriz has $d+12$ beads.

If Diane gives 6 beads to Golriz, then Diane would have $d-6$ beads and Golriz would have $d+12+6=d+18$ beads.

From the given information, $d+18=2(d-6)$. Solving, we have $d+18=2d-12$ or $d=30$.

Thus, Diane has 30 beads and Golriz has $30+12=42$ beads. Therefore, they have $30+42=72$ beads in total.

Problem of the Week

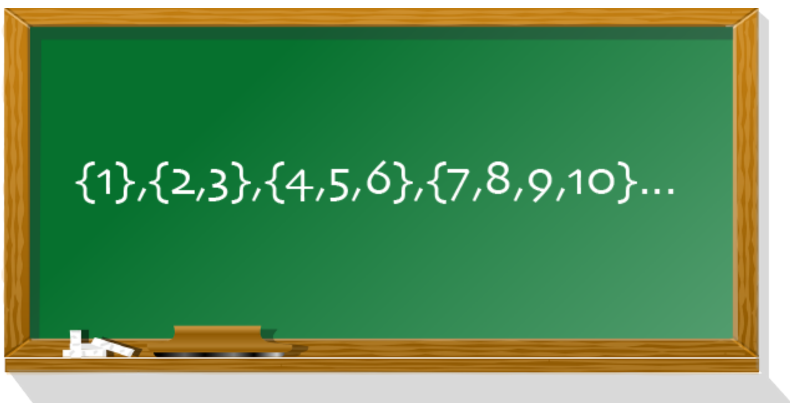
Problem D

Counting Consecutively

Consider the sets of integers $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, $\{7,8,9,10\}$, ...

These sets contain consecutive integers. The first set contains the integer 1, and each set thereafter contains one more integer than the previous set, with its smallest integer being one greater than the largest integer in the previous set.

Determine the sum of the integers in the 101^{st} set.

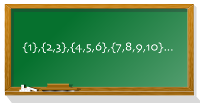


NOTE:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

For example, $1+2+3+4+5=15$, and $\dfrac{5(6)}{2}=15$.

Also, $1+2+3+4+5+6+7+8=36$, and $\dfrac{8(9)}{2}=36$.



Problem of the Week
Problem D and Solution
Counting Consecutively

Problem

Consider the sets of integers $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10\}$, $\ldots$.

These sets contain consecutive integers. The first set contains the integer 1, and each set thereafter contains one more integer than the previous set, with its smallest integer being one greater than the largest integer in the previous set.

Determine the sum of the integers in the 101^{st} set.

NOTE:
In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\dfrac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For example, $1 + 2 + 3 + 4 + 5 = 15$, and $\dfrac{5(6)}{2} = 15$.

Also, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, and $\dfrac{8(9)}{2} = 36$.

Solution
Solution 1

Consider the sequence of the largest numbers in each set: 1, 3, 6, 10, ...

This sequence of numbers consists of *triangular* numbers, which represent the sum of the first n natural numbers. That is, $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$. In general, the n^{th} triangular number, T_n, is the sum of the first n integers from 1 to n. That is, $T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Since each set contains one more integer than the previous set, the largest number in the n^{th} set will be equal to T_n. That is, the largest number in the n^{th} set will be equal to $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Therefore, the largest integer in the 100^{th} set is $T_{100} = \frac{100(100+1)}{2} = 5050$. Thus, the smallest integer in the 101^{st} set is $5050 + 1 = 5051$.

In the 101^{st} set there are 101 integers and the smallest integer is 5051. Therefore, the largest integer is $5050 + 101 = 5151$.

Therefore, sum of the elements in the 101^{st} set is equal to

$$\begin{aligned}
5051 + 5052 + \cdots + 5151 &= (1 + 2 + \cdots + 5151) - (1 + 2 + \cdots 5050) \\
&= \frac{5151(5152)}{2} - \frac{5050(5051)}{2} \\
&= 13\,268\,976 - 12\,753\,775 \\
&= 515\,201
\end{aligned}$$

Solution 2

This solution uses quadratics, which may be beyond what some students have learned.

Consider then the sequence of largest numbers in each set: 1, 3, 6, 10, ...

The first differences of this sequence are 2, 3, 4, ... If we calculate the second differences, each is 1. We will leave it to the reader to justify that the second difference is a constant of 1 for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $f(x)$ is the largest number in the x^{th} set.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 1$, so

$$a + b + c = 1 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 3$, so

$$4a + 2b + c = 3 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 6$, so

$$9a + 3b + c = 6 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 2$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 3$, which we call equation (5).

We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 2$, and so $b = 2 - \frac{3}{2} = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{1}{2}$ into equation (1), we have $\frac{1}{2} + \frac{1}{2} + c = 1$, and so $c = 0$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$.

Therefore, the largest number in the 100^{th} set is $f(100) = \frac{1}{2}(100)^2 + \frac{1}{2}(100) = 5000 + 50 = 5050$.

The largest number in the 101^{st} set is $f(101) = \frac{1}{2}(101)^2 + \frac{1}{2}(101) = 5100.5 + 50.5 = 5151$.

Therefore, sum of the elements in the 101^{st} set is equal to

$$\begin{aligned}
5051 + 5052 + \cdots + 5151 &= (1 + 2 + \cdots + 5151) - (1 + 2 + \cdots 5050) \\
&= \frac{5151(5152)}{2} - \frac{5050(5051)}{2} \\
&= 13\,268\,976 - 12\,753\,775 \\
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\end{aligned}$$

Problem of the Week

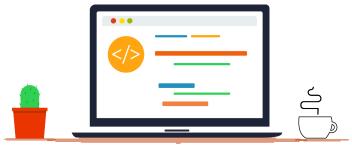
Problem D

One Thousand Inputs

Yeseo has written a program. After she inputs a number into the program the number is multiplied by 3, then 4 is added to the product, then the sum is divided by 3. The final result is then printed.

Yeseo inputs the number 2 into the program. Following the steps outlined, the number 2 is first multiplied by 3 to obtain 6. Then 4 is added to the product to obtain 10. Finally, this sum is divided by 3 to obtain $\frac{10}{3}$. The number $\frac{10}{3}$ is then printed.

Yeseo then inputs the output $\frac{10}{3}$ into the program. If Yeseo continues this process of taking the output number and using it as the next input, then what will be the 1000^{th} number she inputs into the program?



Problem of the Week

Problem D and Solution

One Thousand Inputs

Problem

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Yeseo inputs the number 2 into the program. Following the steps outlined, the number 2 is first multiplied by 3 to obtain 6. Then 4 is added to the product to obtain 10. Finally, this sum is divided by 3 to obtain $\frac{10}{3}$. The number $\frac{10}{3}$ is then printed.

Yeseo then inputs the output $\frac{10}{3}$ into the program. If Yeseo continues this process of taking the output number and using it as the next input, then what will be the 1000^{th} number she inputs into the program?

Solution

One way to solve this problem would be to work through all the different input numbers that Yeseo would have by following the steps in the program. However, since we are looking for the 1000^{th} input number, this is definitely not the most efficient solution.

Instead, we will first determine the first few input numbers. These are shown in the table.

Input	After Multiplying by 3	After Adding 4	After Dividing by 3
2	6	10	$\frac{10}{3}$
$\frac{10}{3}$	10	14	$\frac{14}{3}$
$\frac{14}{3}$	14	18	$\frac{18}{3} = 6$
6	18	22	$\frac{22}{3}$
$\frac{22}{3}$	22	26	$\frac{26}{3}$
$\frac{26}{3}$	26	30	$\frac{30}{3} = 10$

So the first seven input numbers are 2, $\frac{10}{3}$, $\frac{14}{3}$, 6, $\frac{22}{3}$, $\frac{26}{3}$, and 10. It appears as though the input numbers are increasing by $\frac{4}{3}$ each time. We can show this algebraically. Suppose an input number is p. Then following the steps in the program, we first multiply by 3 to obtain $3p$. We then add 4 to obtain $3p + 4$. Finally, we divide by 3 to obtain $\frac{3p+4}{3}$. This can be simplified as follows:

$$\frac{3p+4}{3} = \frac{3p}{3} + \frac{4}{3} = p + \frac{4}{3}$$

Thus, for any input number p, the output (and next input number) will be $p + \frac{4}{3}$.

We will let t_1 be the first input number, t_2 be the second input number, and so on. Thus, we want to find t_{1000}. Using this notation, we now show three different solutions.

Solution 1
Since the difference between consecutive input numbers is $\frac{4}{3}$, then to get from t_1 to t_{1000}, we must add $\frac{4}{3}$ a total of 999 times. Therefore,

$$\begin{aligned} t_{1000} &= t_1 + 999 \times \frac{4}{3} \\ &= 2 + 999 \times \frac{4}{3} \\ &= 2 + 1332 \\ &= 1334 \end{aligned}$$

Thus, the 1000^{th} input number is 1334.

Solution 2
Since the difference between consecutive input numbers is $\frac{4}{3}$, it follows that the difference between every third input number is $3 \times \frac{4}{3} = 4$. Since the first input number, 2, is an integer, it follows that every third input number after that will also be an integer, and that these integer values form a sequence that is increasing by 4 each time. Since $1000 = 1 + 3 \times 333$, then starting from t_1, we must add 4 a total of 333 times to reach t_{1000}. Therefore,

$$\begin{aligned} t_{1000} &= t_1 + 333 \times 4 \\ &= 2 + 333 \times 4 \\ &= 1334 \end{aligned}$$

Thus, the 1000^{th} input number is 1334.

Solution 3
Since the difference between consecutive input numbers is $\frac{4}{3}$, if all input numbers are written with a denominator of 3, then the difference between the numerators of consecutive input numbers is 4. Thus, we can think of the numerators of consecutive input numbers as a sequence that starts at 6 (since $2 = \frac{6}{3}$) and increases by 4 each time. This sequence can be written as $4n + 2$, where $n = 1$ represents the first input number, $n = 2$ represents the second input number, and so on. Thus, $t_n = \frac{4n+2}{3}$ for any $n \geq 1$. Therefore,

$$t_{1000} = \frac{4(1000) + 2}{3} = \frac{4002}{3} = 1334$$

Thus, the 1000^{th} input number is 1334.

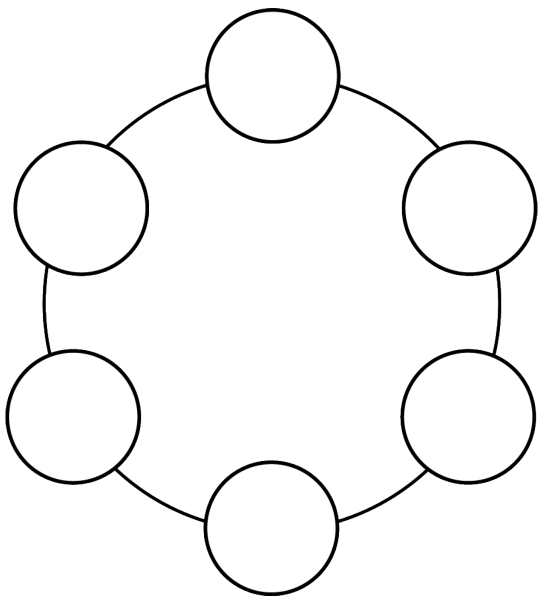
Problem of the Week

Problem D

Number Rings

The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.



Problem of the Week
Problem D and Solution
Number Rings

Problem

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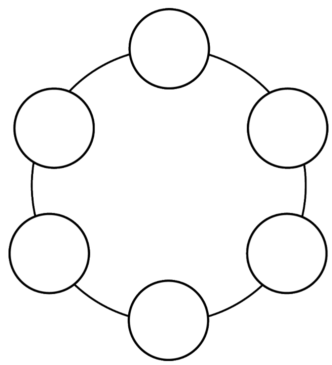


Solution

For each number, we first determine which numbers it can be adjacent to:

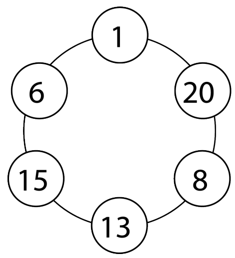
- Number 1: 6, 13, 20
- Number 6: 1, 8, 15
- Number 8: 6, 13, 20
- Number 13: 1, 8, 15
- Number 15: 6, 13, 20
- Number 20: 1, 8, 15

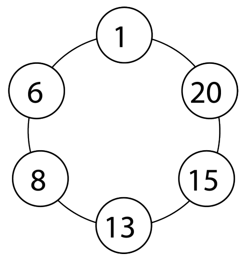
We now determine all the different arrangements by considering various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Consider the possibilities for the numbers adjacent to 1. There are three possible cases: 1 is adjacent to 6 and 20, 1 is adjacent to 6 and 13, and 1 is adjacent to 13 and 20. We consider each case separately.

Case 1: 1 is adjacent to 6 and 20

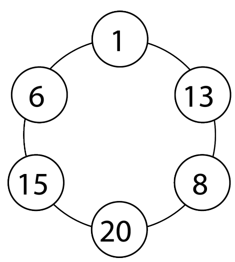
In this case, 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

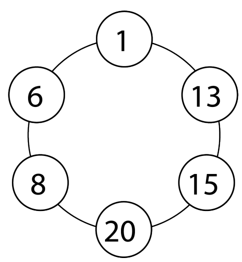




Case 2: 1 is adjacent to 6 and 13

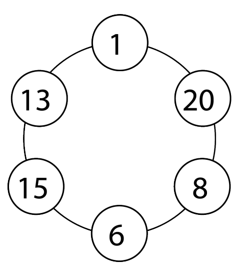
In this case, 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

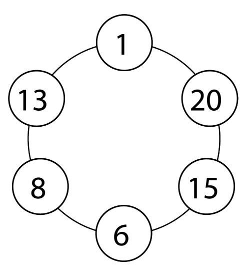




Case 3: 1 is adjacent to 13 and 20

In this case, 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.





Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2, and 3 above.