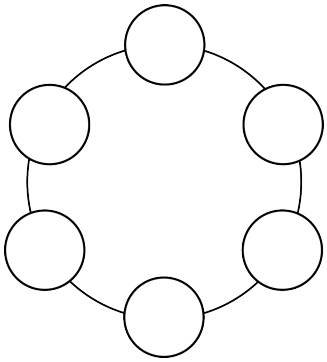
Problem of the Week
Problem D and Solution
Number Rings

Problem

The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.



Solution

For each number, we first determine which numbers it can be adjacent to:

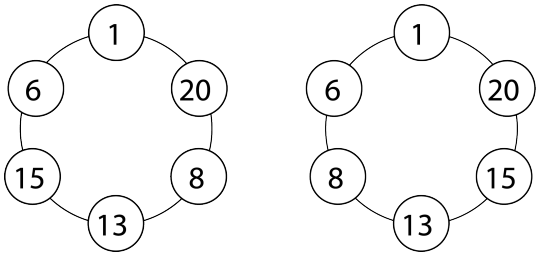
- Number 1: 6, 13, 20
- Number 6: 1, 8, 15
- Number 8: 6, 13, 20
- Number 13: 1, 8, 15
- Number 15: 6, 13, 20
- Number 20: 1, 8, 15

We now determine all the different arrangements by considering various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Consider the possibilities for the numbers adjacent to 1. There are three possible cases: 1 is adjacent to 6 and 20, 1 is adjacent to 6 and 13, and 1 is adjacent to 13 and 20. We consider each case separately.

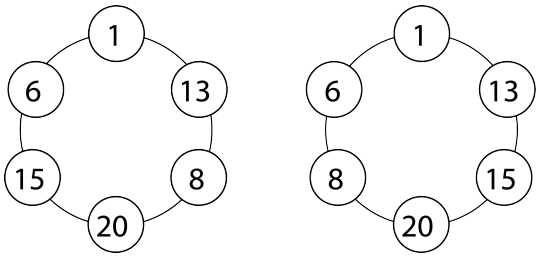
Case 1: 1 is adjacent to 6 and 20

In this case, 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.



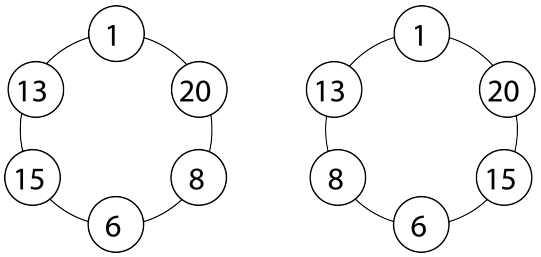
Case 2: 1 is adjacent to 6 and 13

In this case, 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.



Case 3: 1 is adjacent to 13 and 20

In this case, 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.



Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2, and 3 above.