Problem of the Week
Problem D and Solution
Cheesecake Geometry

Problem

For Amanda's birthday, Rhett made a chocolate cheesecake. The cake was in the shape of a cylinder, with equal radius and height. Rhett cut the cake into 8 congruent slices, each in the shape of a sector of a cylinder. Rhett then ate one slice for quality control purposes.

After removing Rhett's slice, by what percentage has the surface area of the cake increased or decreased? Round your answer to one decimal place.

Solution

Let the cake have radius r and height h. We are given that $r = h$.

We first calculate the surface area of the entire cake without the slice removed.

The surface area of a cylinder includes the areas of the two circular faces plus the surface area of the side. The circular faces each have area equal to πr^2. Since the circumference of a circular face is $2\pi r$, the surface area of the side is $(2\pi r)(h) = 2\pi r^2$, since $r = h$. Thus, the total surface area of the cake before the slice is removed is $\pi r^2 + \pi r^2 + 2\pi r^2 = 4\pi r^2$.

We now calculate the surface area of the cake after the slice is removed.

The surface area removed includes $\frac{1}{8}$ of the area of each of the two circular faces with radius r and $\frac{1}{8}$ of the surface area of the side of the cylinder. Therefore, the surface area remaining includes $1 - \frac{1}{8} = \frac{7}{8}$ of the original surface area. That is, $\frac{7}{8} \times 4\pi r^2 = \frac{7\pi r^2}{2}$.

There are two areas that are added as a result of removing the slice. These areas are the rectangular faces of the slice that were inside the cylindrical cake before the slice was removed. These rectangles each have length equal to the radius of the circular face, r, and width $h = r$. Thus, the surface area added is equal to $2(r)(r) = 2r^2$.

Therefore, the new total surface area is equal to $\frac{7\pi r^2}{2} + 2r^2$.

The change in surface area is $(\frac{7\pi r^2}{2} + 2r^2) - 4\pi r^2 = -\frac{\pi r^2}{2} + 2r^2 = r^2(2 - \frac{\pi}{2})$.

Now, $\frac{\pi}{2} < 2$, so $2 - \frac{\pi}{2} > 0$. Therefore, the surface area increases as a result of removing the slice.

To calculate the percentage that the area has increased, we divide the surface area increase by the original surface area, and multiply by 100%.

The increase is $r^2 \left(2 - \frac{\pi}{2}\right)$. Therefore, the percentage increase in surface area is equal to

$$\frac{r^2 \left(2 - \frac{\pi}{2}\right)}{4\pi r^2} \times 100\% = \frac{\left(2 - \frac{\pi}{2}\right)}{4\pi} \times 100\%$$
$$= \left(\frac{4 - \pi}{8\pi}\right) \times 100\%$$
$$\approx 3.4\%$$

Therefore, the surface area of the cake increases by approximately 3.4% after the slice has been removed.