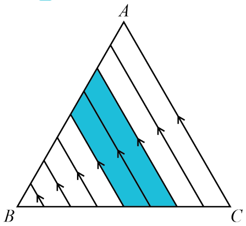


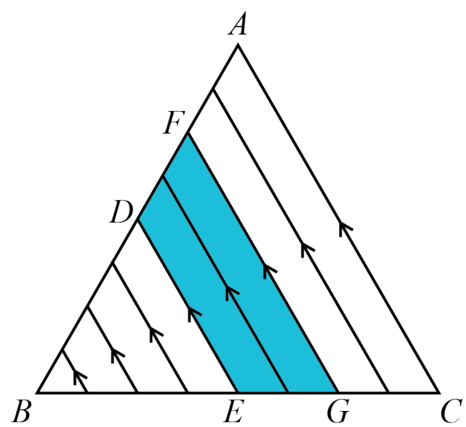
Problem of the Week
Problem D and Solution
Another Suncatcher

Problem
A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

- Sides AB and BC are each divided into 8 segments of equal length.
- Each point of division on AB is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

Two of the sections are coloured blue to form a trapezoid, as shown. Determine the area of this trapezoid.

Solution
Solution 1
We start by labeling the vertices of the trapezoid D, E, F, and G, as shown. In this solution we will subtract the area of $\triangle BDE$ from the area of $\triangle BFG$ to find the area of trapezoid $DEGF$.



Since AB and BC are each divided into 8 equal segments, each of these segments must have length $144 \div 8 = 18$ mm. Since BD is made up of 4 of these segments, $BD = 4 \times 18 = 72$ mm. Similarly, since BF is made up of 6 of these segments, $BF = 6 \times 18 = 108$ mm.

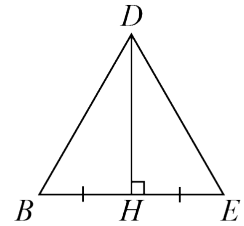
First we will show that $\triangle BDE$ and $\triangle BFG$ are equilateral triangles. Since $\triangle ABC$ is equilateral, $\angle BAC = \angle ABC = \angle ACB = 60^\circ$. Since $\angle DBE$, $\angle FBG$ and $\angle ABC$ are the same angle, $\angle DBE = \angle FBG = \angle ABC = 60^\circ$.

Since $DE \parallel FG \parallel AC$, $\angle BED = \angle BGF = \angle BCA = 60^\circ$ and $\angle BDE = \angle BFG = \angle BAC = 60^\circ$.

Since each angle in $\triangle BDE$ and $\triangle BFG$ is 60°, both triangles are equilateral. $\triangle BDE$ has side length 72 mm and $\triangle BFG$ has side length 108 mm.

Next we need to determine the height of each triangle. In $\triangle BDE$, drop a perpendicular from D to H on BE. Since $\triangle BDE$ is equilateral, H is the midpoint of BE and it follows that $BH = \frac{1}{2}BE = \frac{1}{2}(72) = 36$ mm.

Using the Pythagorean Theorem in $\triangle BHD$,



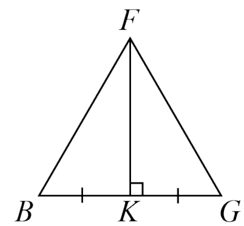
$$\begin{aligned} DH^2 &= BD^2 - BH^2 \\ &= 72^2 - 36^2 = 3888 \end{aligned}$$

Therefore $DH = \sqrt{3888} = 36\sqrt{3}$ mm, since $DH \geq 0$.

Therefore, the area of $\triangle BDE$ is $\frac{(BE)(DH)}{2} = \frac{(72)(36\sqrt{3})}{2} = 1296\sqrt{3}$ mm^2.

In $\triangle BFG$, drop a perpendicular from F to K on BG. Since $\triangle BFG$ is equilateral, K is the midpoint of BG and it follows that $BK = \frac{1}{2}BG = \frac{1}{2}(108) = 54$ mm.

Using the Pythagorean Theorem in $\triangle BKF$,



$$\begin{aligned} FK^2 &= BF^2 - BK^2 \\ &= 108^2 - 54^2 = 8748 \end{aligned}$$

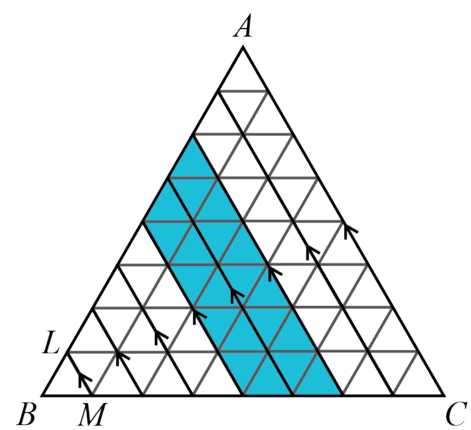
Therefore $FK = \sqrt{8748} = 54\sqrt{3}$ mm, since $FK \geq 0$.

Therefore, the area of $\triangle BFG$ is $\frac{(BG)(FK)}{2} = \frac{(108)(54\sqrt{3})}{2} = 2916\sqrt{3}$ mm^2. Thus, the area of trapezoid $DEGF$ is area $\triangle BFG -$ area $\triangle BDE = 2916\sqrt{3} - 1296\sqrt{3} = 1620\sqrt{3}$ mm^2.

Note: We could also have found the lengths of DH and FK by recognizing that $\triangle BHD$ and $\triangle BKF$ are $30^\circ - 60^\circ - 90^\circ$ triangles with side lengths in the ratio $1 : \sqrt{3} : 2$.

Solution 2

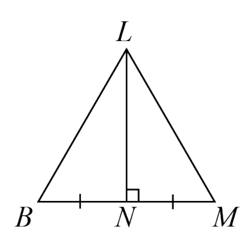
Label the endpoints of the line segment closest to B as L and M. Observe that $\triangle ABC$ can be tiled with small equilateral triangles congruent to $\triangle BLM$. That is, equilateral triangles with side length 18 mm. A complete justification of this is not provided here but you may wish to verify this for yourself.



In the second section from B there are 3 small equilateral triangles. In the third section there are 5 small equilateral triangles. In the fourth section there are 7 small equilateral triangles, and so on. The shaded region contains 20 small equilateral triangles. To find the area of the shaded region, we will find the area of the small equilateral triangle.

In $\triangle BLM$, drop a perpendicular from L to N on BM. Since $\triangle BLM$ is equilateral, N is the midpoint of BM and it follows that $BN = \frac{1}{2}BM = \frac{1}{2}(18) = 9$ mm.

Using the Pythagorean Theorem in $\triangle BLN$,



$$LN^2 = BL^2 - BN^2 = 18^2 - 9^2 = 243$$

Therefore $LN = \sqrt{243} = 9\sqrt{3}$ mm, since $LN \geq 0$.

Therefore, the area of $\triangle BLM$ is $\frac{(BM)(LN)}{2} = \frac{(18)(9\sqrt{3})}{2} = 81\sqrt{3}$ mm^2. Thus, the area of the blue trapezoid is $20 \times 81\sqrt{3} = 1620\sqrt{3}$ mm^2.