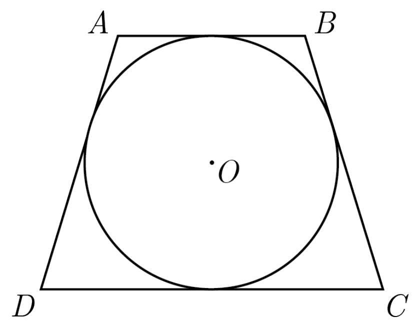
Problem of the Week

Problem D and Solution

A Circle in a Trapezoid

Problem

A circle with centre O and radius 15 m is inside trapezoid $ABCD$ such that each side of $ABCD$ is tangent to the circle. In the trapezoid, $AB \parallel CD$ and $AD = BC$, so $ABCD$ is an isosceles trapezoid.



If the area of $ABCD$ is 2025 m^2, determine the lengths of AD and BC.

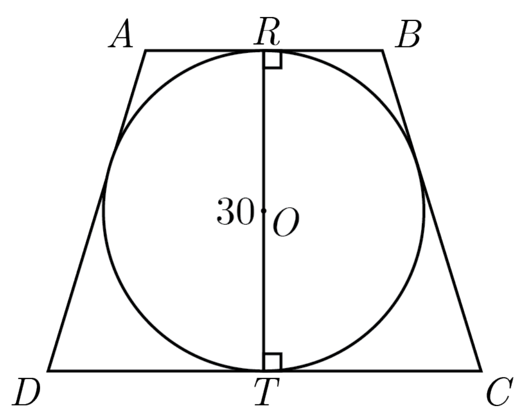
Note: For this problem, you may want to use the following known results about circles:

1. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.
2. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.

Solution

Draw line segment RT through O perpendicular to AB and CD so that R lies on AB and T lies on CD. From Result 2 given after the problem, R and T are the points of tangency on AB and CD, respectively.

Since OR and OT are both radii of the circle, $RT = OR + OT = 15 + 15 = 30$. Since $RT \perp CD$, RT is the height of trapezoid $ABCD$.



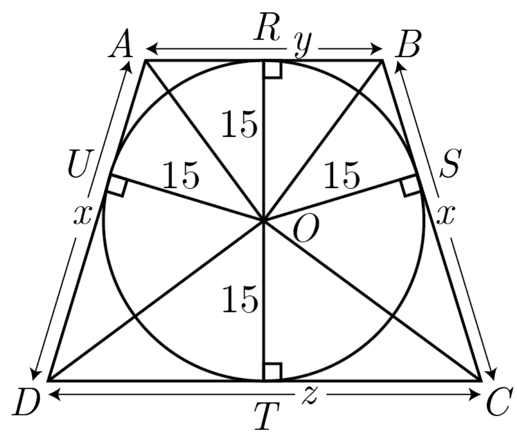
We will now proceed with two different approaches.

Solution 1

Let $y = AB$ and $z = CD$. Then using the formula for the area of a trapezoid,

$$\begin{aligned} \text{Area} &= (AB + CD) \times (RT) \div 2 \\ 2025 &= (y + z) \times 30 \div 2 \\ 2025 &= (y + z) \times 15 \\ 135 &= y + z \qquad (1) \end{aligned}$$

Let $x = AD = BC$. Join O to each of the vertices of $ABCD$, creating four triangles, $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$. Then join O to each of the points of tangency, R, S, T, and U, on AB, BC, CD, and AD, respectively. Each of these line segments is a radius so $OR = OS = OT = OU = 15$. Furthermore, from Result 1 given after the problem, these line segments are each perpendicular to their respective sides of the trapezoid.

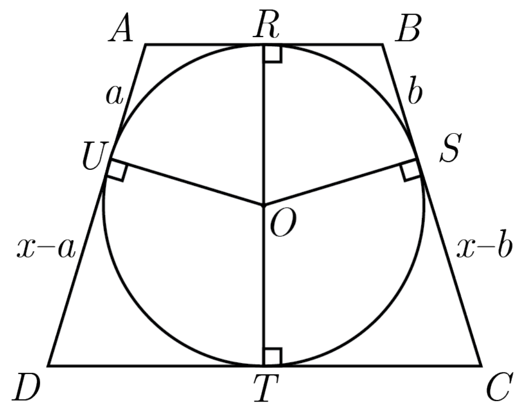


We can now find the area of the trapezoid a second way by summing the areas of the four triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$.

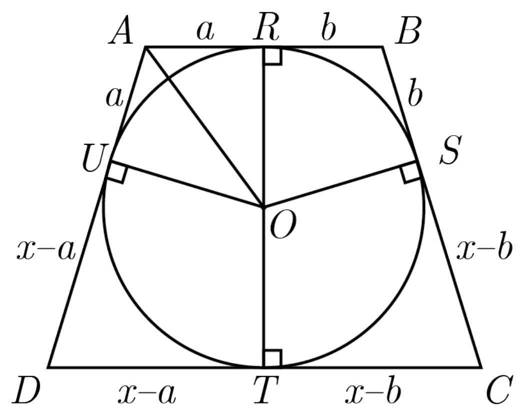
$$\begin{aligned} \text{Area} &= \text{Area } \triangle AOB + \text{Area } \triangle BOC + \text{Area } \triangle COD + \text{Area } \triangle AOD \\ 2025 &= \frac{AB \times OR}{2} + \frac{BC \times OS}{2} + \frac{CD \times OT}{2} + \frac{AD \times OU}{2} \\ 2025 &= \frac{15y}{2} + \frac{15x}{2} + \frac{15z}{2} + \frac{15x}{2} \\ 2025 &= 15x + \frac{15(y+z)}{2} \\ 2025 &= 15x + \frac{15(135)}{2} \qquad \text{(from (1))} \\ 2025 &= 15x + \frac{2025}{2} \\ 15x &= \frac{2025}{2} \\ x &= \frac{135}{2} = 67.5 \end{aligned}$$

Therefore, AD and BC each have a length of 67.5 m.

Solution 2



Join O to each of the points of tangency, R, S, T, and U, on AB, BC, CD, and AD, respectively. From Result 1 given after the problem, these line segments are each perpendicular to their respective sides of the trapezoid.
Let $x = AD = BC$, $a = AU$ and $b = BS$. Then $DU = x - a$ and $CS = x - b$.



Join O to A, forming right-angled triangles $\triangle AUO$ and $\triangle ARO$. Using the Pythagorean theorem, $AO^2 = AU^2 + OU^2$ and $AO^2 = AR^2 + OR^2$. Since OU and OR are both radii, $OU = OR$. Thus $AR = AU = a$. Using a similar reasoning, we can conclude that $BR = BS = b$, $CT = CS = x - b$, and $DT = DU = x - a$.

Then using the formula for the area of a trapezoid,

$$\begin{aligned}
\text{Area} &= (AB + CD) \times (RT) \div 2 \\
2025 &= ((AR + BR) + (DT + CT)) \times 30 \div 2 \\
2025 &= ((a + b) + (x - a + x - b) \times 15 \\
2025 &= 2x \times 15 \\
2025 &= 30x \\
x &= \frac{2025}{30} = 67.5
\end{aligned}$$

Therefore, AD and BC each have a length of 67.5 m.