Problem of the Week
Problem D and Solution
Favourite Number 2

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Ayumi announced that her favourite number is a three-digit positive integer with a digit sum of 9. Vinnie then wrote down a three-digit positive integer with a digit sum of 9. What is the probability that the number Vinnie wrote was Ayumi's favourite number?

Solution

There are twelve triples of digits that sum to 9. These are: $(9, 0, 0)$, $(8, 1, 0)$, $(7, 2, 0)$, $(7, 1, 1)$, $(6, 3, 0)$, $(6, 2, 1)$, $(5, 4, 0)$, $(5, 3, 1)$, $(5, 2, 2)$, $(4, 4, 1)$, $(4, 3, 2)$, and $(3, 3, 3)$.

We will count the number of three-digit positive integers whose digit sum is 9 by counting the number of positive integers formed by the digits in each triple above. We will consider cases based on the number of zeroes in each triple, since a three-digit integer cannot have a leading digit of 0.

- **Case 1:** Two zeroes
  The only triple with two zeros is $(9, 0, 0)$. Since the first digit of the three-digit integer cannot be zero, the only possible three-digit integer is 900. Thus, there is only 1 possible three-digit integer with two zeroes.

- **Case 2:** One zero
  The triples with one zero are $(8, 1, 0)$, $(7, 2, 0)$, $(6, 3, 0)$, and $(5, 4, 0)$. Consider $(8, 1, 0)$. Since the zero cannot be in the first position, there are four integers that can be formed with these three digits: 810, 801, 180, and 108. Since this is true for any of the four triples in this case, there are $4 \times 4 = 16$ possible three-digit integers with one zero.

- **Case 3:** No zeroes
  In this case, we will further separate the triples based on the number of repeated digits.

  - **Case 3a:** Three repeated digits
    The only triple with three repeated digits is $(3, 3, 3)$ and the only integer formed by these digits is 333. Thus, there is only 1 possible three-digit integer in this case.

- **Case 3b:** Two repeated digits
  The triples with two repeated digits are $(7, 1, 1)$, $(5, 2, 2)$, and $(4, 4, 1)$. Consider $(7, 1, 1)$. There are three integers that can be formed by these three digits: 711, 171, and 117. Since this is true for any of the three triples in this case, there are $3 \times 3 = 9$ possible three-digit integers in this case.

- **Case 3c:** No repeated digits
  The triples are $(6, 2, 1)$, $(5, 3, 1)$, and $(4, 3, 2)$. Consider $(6, 2, 1)$. There are six integers that can be formed by these three digits: 621, 612, 261, 216, 162, and 126. Since this is true for any of the three triples in this case, there are $3 \times 6 = 18$ possible three-digit integers in this case.

Therefore, there are $1 + 9 + 18 = 28$ possible three-digit integers with no zeroes.

Therefore, the total number of three-digit positive integers that have a digit sum of 9 is $1 + 16 + 28 = 45$. Since Vinnie wrote down only one of these integers, the probability that this integer was Ayumi's favourite number is $\frac{1}{45}$.

NOTE: It is a known fact that an integer is divisible by 9 exactly when its digit sum is divisible by 9. For example, $32\,814$ has a digit sum of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 9, then $32\,814$ is divisible by 9. On the other hand, $32\,810$ has a digit sum of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 9, then $32\,810$ is not divisible by 9.

As a consequence of this fact, Ayumi's favourite number must be divisible by 9.