



Problem of the Week

Problem D and Solution

Counting Consecutively

Problem

Consider the sets of integers $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10\}$, \dots

These sets contain consecutive integers. The first set contains the integer 1, and each set thereafter contains one more integer than the previous set, with its smallest integer being one greater than the largest integer in the previous set.

Determine the sum of the integers in the 101st set.

NOTE:

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

For example, $1 + 2 + 3 + 4 + 5 = 15$, and $\frac{5(6)}{2} = 15$.

Also, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, and $\frac{8(9)}{2} = 36$.

Solution

Solution 1

Consider the sequence of the largest numbers in each set: 1, 3, 6, 10, \dots

This sequence of numbers consists of *triangular* numbers, which represent the sum of the first n natural numbers. That is, $1 = 1$, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$. In general, the n^{th} triangular number, T_n , is the sum of the first n integers from 1 to n . That is, $T_n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

Since each set contains one more integer than the previous set, the largest number in the n^{th} set will be equal to T_n . That is, the largest number in the n^{th} set will be equal to $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

Therefore, the largest integer in the 100th set is $T_{100} = \frac{100(100+1)}{2} = 5050$. Thus, the smallest integer in the 101st set is $5050 + 1 = 5051$.

In the 101st set there are 101 integers and the smallest integer is 5051. Therefore, the largest integer is $5050 + 101 = 5151$.

Therefore, sum of the elements in the 101st set is equal to

$$\begin{aligned} 5051 + 5052 + \dots + 5151 &= (1 + 2 + \dots + 5151) - (1 + 2 + \dots + 5050) \\ &= \frac{5151(5152)}{2} - \frac{5050(5051)}{2} \\ &= 13\,268\,976 - 12\,753\,775 \\ &= 515\,201 \end{aligned}$$



Solution 2

This solution uses quadratics, which may be beyond what some students have learned.

Consider then the sequence of largest numbers in each set: 1, 3, 6, 10, ...

The first differences of this sequence are 2, 3, 4, ... If we calculate the second differences, each is 1. We will leave it to the reader to justify that the second difference is a constant of 1 for the entire sequence.

This tells us that $f(x)$ is quadratic. That is, $f(x) = ax^2 + bx + c$, where $f(x)$ is the largest number in the x^{th} set.

We know that $f(1) = a(1)^2 + b(1) + c = a + b + c$ and $f(1) = 1$, so

$$a + b + c = 1 \tag{1}$$

We know that $f(2) = a(2)^2 + b(2) + c = 4a + 2b + c$ and $f(2) = 3$, so

$$4a + 2b + c = 3 \tag{2}$$

We know that $f(3) = a(3)^2 + b(3) + c = 9a + 3b + c$ and $f(3) = 6$, so

$$9a + 3b + c = 6 \tag{3}$$

We first eliminate c from the system of equations. We subtract equation (1) from equation (2) to obtain $3a + b = 2$, which we call equation (4). Then, we subtract equation (2) from equation (3) to obtain $5a + b = 3$, which we call equation (5).

We now eliminate b from the system of equations. We subtract equation (4) from equation (5), to obtain $2a = 1$, and so $a = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ into equation (4), we have $3\left(\frac{1}{2}\right) + b = 2$, and so $b = 2 - \frac{3}{2} = \frac{1}{2}$.

Substituting $a = \frac{1}{2}$ and $b = \frac{1}{2}$ into equation (1), we have $\frac{1}{2} + \frac{1}{2} + c = 1$, and so $c = 0$.

Therefore, the quadratic function is $f(x) = \frac{1}{2}x^2 + \frac{1}{2}x$.

Therefore, the largest number in the 100th set is $f(100) = \frac{1}{2}(100)^2 + \frac{1}{2}(100) = 5000 + 50 = 5050$.

The largest number in the 101st set is $f(101) = \frac{1}{2}(101)^2 + \frac{1}{2}(101) = 5100.5 + 50.5 = 5151$.

Therefore, sum of the elements in the 101st set is equal to

$$\begin{aligned} 5051 + 5052 + \cdots + 5151 &= (1 + 2 + \cdots + 5151) - (1 + 2 + \cdots + 5050) \\ &= \frac{5151(5152)}{2} - \frac{5050(5051)}{2} \\ &= 13\,268\,976 - 12\,753\,775 \\ &= 515\,201 \end{aligned}$$