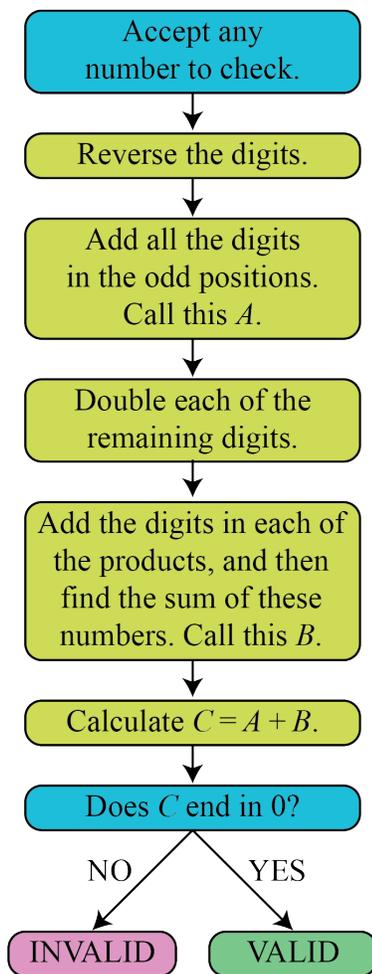# Problem of the Week
## Problem D and Solution
## Check Please 2

**Problem**

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

Accept any number to check.

↓

Reverse the digits.

↓

Add all the digits in the odd positions. Call this $A$.

↓

Double each of the remaining digits.

↓

Add the digits in each of the products, and then find the sum of these numbers. Call this $B$.

↓

Calculate $C = A + B$.

↓

Does $C$ end in 0?

NO → INVALID    YES → VALID

**Example 1**

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: 2̲97̲53̲1
$$A = 2 + 7 + 3$$
$$= 12$$
- Double remaining digits:
$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$
- Calculate $B$:
$$B = (1 + 8) + (1 + 0) + 2$$
$$= 9 + 1 + 2$$
$$= 12$$
- $C = 12 + 12 = 24$
- $C$ does not end in zero, so the number is not valid.

**Example 2**

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: 7̲89̲75̲31̲
$$A = 7 + 9 + 5 + 1$$
$$= 22$$
- Double remaining digits:
$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$
- Calculate $B$:
$$B = (1 + 6) + (1 + 4) + 6$$
$$= 7 + 5 + 6$$
$$= 18$$
- $C = 22 + 18 = 40$
- $C$ ends in zero, so the number is valid.

The number 8764 $X6X5$ $X6X8$ 5409 is a valid card number when verified by the Luhn Algorithm, where $X$ is a single digit. Determine all possible values of $X$.

**Solution**

**Solution 1**

When the digits of the number are reversed the resulting number is
9045 $8X6X$ $5X6X$ 4678. The sum of the digits in the odd positions is

$$A = 9 + 4 + 8 + 6 + 5 + 6 + 4 + 7 = 49.$$

When the digits in the remaining positions are doubled, the following products
are obtained:

$$2(0) = 0; \ 2(5) = 10; \ 2(X) = 2X; \ 2(X) = 2X;$$

$$2(X) = 2X; \ 2(X) = 2X; \ 2(6) = 12; \ 2(8) = 16$$

Let $n$ represent the sum of the digits of $2X$.

When the digit sums from each of the products are added, the sum is:

$$B = 0 + (1 + 0) + n + n + n + n + (1 + 2) + (1 + 6) = 0 + 1 + 4n + 3 + 7 = 4n + 11.$$

Since $C = A + B$, we have $C = 49 + 4n + 11 = 60 + 4n$.

When an integer from 0 to 9 is doubled and the digits of the product are added
together, what possible sums can be obtained? We summarize these in the
following table.

| Original Digit $(X)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Twice the Original Digit $(2X)$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| The Sum of the Digits of $2X$ $(n)$ | 0 | 2 | 4 | 6 | 8 | 1 | 3 | 5 | 7 | 9 |

Notice that the sum of the digits of $2X$ can be any integer from 0 to 9 inclusive.
It follows that the only values for $n$ are the integers from 0 to 9.

To be a valid number, the units digit of $C$ must be 0. We want $60 + 4n$ to be an
integer whose units digit is 0. Since the minimum value of $n$ is 0, it follows that
$60 + 4n \geq 60$. We will now look at cases based on the value of $60 + 4n$.

- Case 1: $60 + 4n = 60$.
  Then $4n = 0$, so $n = 0$. From our table, this occurs when $X = 0$. This is
  one possible value for $X$.

- Case 2: $60 + 4n = 70$.
  Then $4n = 10$, so $n = 2.5$. However $n$ must be an integer value so this is not
  possible.

- Case 3: $60 + 4n = 80$.
  Then $4n = 20$, so $n = 5$. From our table, this occurs when $X = 7$. This is one possible value for $X$.

- Case 4: $60 + 4n = 90$.
  Then $4n = 30$, so $n = 7.5$. However $n$ must be an integer value so this is not possible.

- Case 5: $60 + 4n = 100$.
  Then $4n = 40$, so $n = 10$. However $n$ must be an integer from 0 to 9 inclusive, so this is not possible. This tells us that $60 + 4n < 100$, so there are no further cases.

Therefore, the two valid possibilities for $X$ are 0 and 7.

When $X = 0$, the number is 8764 0605 0608 5409, which is indeed valid by the Luhn Algorithm.
When $X = 7$, the number is 8764 7675 7678 5409 which is indeed valid by the Luhn Algorithm.

## Solution 2
The second solution looks at each of the possible values of $X$ and then verifies the resulting number. A computer program or spreadsheet could be developed to solve this problem efficiently.

Remember that $A$ is the sum of the digits in the odd positions of the reversal. Each of the digits in the even positions of the reversal are doubled, and $B$ is the sum of the sum of the digits of each of these products. $C$ is the sum $A + B$.

| $X$ | Number | Reversal | $A$ | Double Even Digits | $B$ | $C$ | Valid? |
|---|---|---|---|---|---|---|---|
| 0 | 8764 0605 0608 5409 | 9045 8060 5060 4678 | 49 | 0, 10, 0, 0, 0, 0, 12, 16 | 11 | 60 | Yes |
| 1 | 8764 1615 1618 5409 | 9045 8161 5161 4678 | 49 | 0, 10, 2, 2, 2, 2, 12, 16 | 19 | 68 | No |
| 2 | 8764 2625 2628 5409 | 9045 8262 5262 4678 | 49 | 0, 10, 4, 4, 4, 4, 12, 16 | 27 | 76 | No |
| 3 | 8764 3635 3638 5409 | 9045 8363 5363 4678 | 49 | 0, 10, 6, 6, 6, 6, 12, 16 | 35 | 84 | No |
| 4 | 8764 4645 4648 5409 | 9045 8464 5464 4678 | 49 | 0, 10, 8, 8, 8, 8, 12, 16 | 43 | 92 | No |
| 5 | 8764 5655 5658 5409 | 9045 8565 5565 4678 | 49 | 0, 10, 10, 10, 10, 10, 12, 16 | 15 | 64 | No |
| 6 | 8764 6665 6668 5409 | 9045 8666 5666 4678 | 49 | 0, 10, 12, 12, 12, 12, 12, 16 | 23 | 72 | No |
| 7 | 8764 7675 7678 5409 | 9045 8767 5767 4678 | 49 | 0, 10, 14, 14, 14, 14, 12, 16 | 31 | 80 | Yes |
| 8 | 8764 8685 8688 5409 | 9045 8868 5868 4678 | 49 | 0, 10, 16, 16, 16, 16, 12, 16 | 39 | 88 | No |
| 9 | 8764 9695 9698 5409 | 9045 8969 5969 4678 | 49 | 0, 10, 18, 18, 18, 18, 12, 16 | 47 | 96 | No |

Therefore, the two valid possibilities for $X$ are 0 and 7.