Problem of the Week

Problem D and Solution

Number Triangle

Problem

Consecutive positive integers are arranged in rows into the shape of a triangle. In this triangle, the top row contains the integer 1, and each row below the top row contains one more integer than the row above, starts with the next consecutive integer after the largest integer in the row above, and lists the consecutive integers in increasing order. That is, the first row contains the integer 1, the second row contains the integers 2 and 3, the third row contains the integers 4, 5, and 6, the fourth row contains the integers 7, 8, 9, and 10, and so on.

In which row will the integer 2026 appear? What integers are in this row?

1

2 3

4 5 6

7 8 9 10

11 12 13 14 15

⋮

In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution

After the first row, there is 1 integer in the triangle.

After the second row, there are $1 + 2 = 3$ integers in the triangle.

After the third row, there are $1 + 2 + 3 = 6$ integers in the triangle.

After the fourth row, there are $1 + 2 + 3 + 4 = 10$ integers in the triangle.

After the fifth row, there are $1 + 2 + 3 + 4 + 5 = 15$ integers in the triangle.

After the n^{th} row, there are $1 + 2 + 3 + \cdots + n$ integers in the triangle.

If 2026 is in row n, then $1 + 2 + 3 + \cdots + n \geq 2026$ and $1 + 2 + 3 + \cdots + (n-1) < 2026$.

To determine n, let's try solving $1 + 2 + \cdots + n = 2026$.

It is given that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Thus, we are looking to solve $\frac{n(n+1)}{2} = 2026$, or equivalently $n^2 + n = 4052$.

At this point we will use trial and error. At the end of the solution, a more algebraic approach using the quadratic formula is presented.

Suppose $n = 25$. Then $n^2 + n = 25^2 + 25 = 650 < 4052$.

We can try a larger value for n. Suppose $n = 50$. Then, $n^2 + n = 50^2 + 50 = 2550 < 4052$.

We can try another yet larger value for n. Suppose $n = 65$. Then,
$n^2 + n = 65^2 + 65 = 4290 > 4052$.

We are now close. We can try a smaller value for n. Suppose $n = 63$. Then,
$n^2 + n = 63^2 + 63 = 4032 < 4052$.

Since this value is very close to 4052, we can then try the next integer, so let $n = 64$. Then,
$n^2 + n = 64^2 + 64 = 4160 > 4052$.

This tells us that the integer 2026 is in the 64^{th} row. Further, the first 63 rows contain the integers from 1 to $\frac{63(64)}{2} = 2016$. Therefore, the 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.

We will finish by showing how we can find the value of n algebraically. We will first find the value of $n, n > 0$, so that

$$\begin{aligned} \frac{n(n+1)}{2} &= 2026 \\ n(n+1) &= 4052 \\ n^2 + n - 4052 &= 0 \end{aligned}$$

The quadratic formula can be used to solve for n.

$$\begin{aligned} n &= \frac{-1 \pm \sqrt{1 - 4(1)(-4052)}}{2} \\ n &= \frac{-1 \pm \sqrt{16\,209}}{2} \end{aligned}$$

It follows that $n \approx 63.157$ or $n \approx -64.157$. Since $n \approx -64.157 < 0$, it is inadmissible. Then $n \approx 63.157$.

Notice that $1 + 2 + 3 + \cdots + 63 = \frac{63(64)}{2} = 2016 < 2026$, and
$1 + 2 + 3 + \cdots + 64 = \frac{64(65)}{2} = 2080 \geq 2026$.

Thus, the integer 2026 is in the 64^{th} row. The 64^{th} row contains 64 integers, starting with the integer 2017. That is, it contains the integers from 2017 to 2080, inclusive.