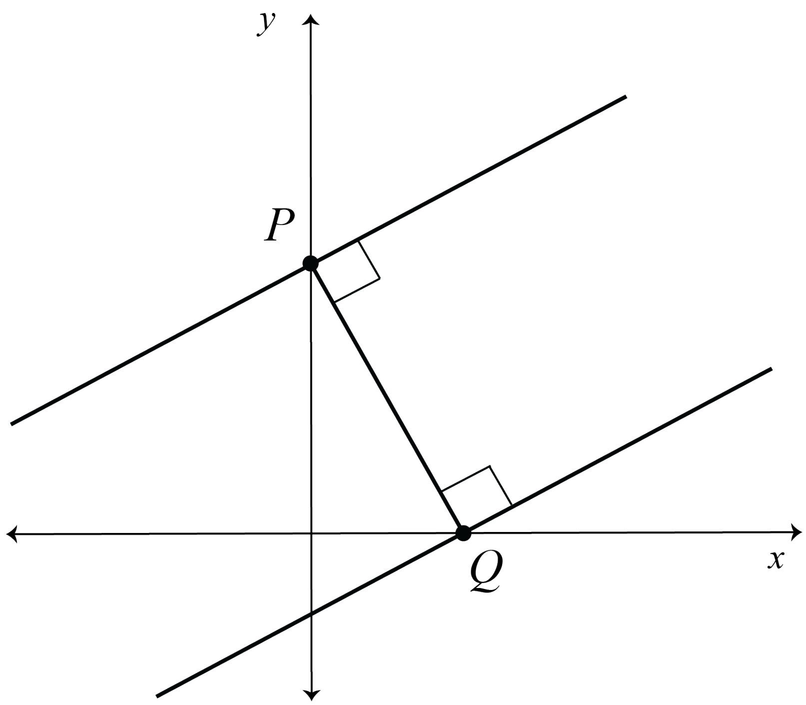
Problem of the Week
Problem D and Solution
Parallel Puzzle

Problem
Two distinct lines are drawn such that the first line passes through point P on the positive y-axis and the second line passes through point Q on the positive x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



Suggestion: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

Solution
We let l_1 represent the first line, which passes through P, and let l_2 represent the second line, which passes through Q.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P has coordinates $(0, k)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{m}$. Since k is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{m}x + k$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of the line through PQ is mk and Q has coordinates $(mk, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is m. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = mx + b$. Since $Q(mk, 0)$ lies on l_2, we have $0 = m(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2. A solution follows.

Let l_1 represent the line with equation $y = 4x + 3$. Let l_2 represent the second line, which passes through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P has coordinates $(0, 3)$.

Since PQ is perpendicular to l_1, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, the slope of PQ is $-\frac{1}{4}$. Since 3 is the y-intercept of segment PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-coordinate of Q is equal to the x-intercept of the line with equation $y = -\frac{1}{4}x + 3$. To determine the x-intercept, we set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result $x = 12$ follows. Therefore, the x-intercept of the line through PQ is 12 and Q has coordinates $(12, 0)$.

Since PQ is perpendicular to l_2 and perpendicular to l_1, it follows that l_2 is parallel to l_1. Thus, the slope of l_2 is 4. Let b represent the y-intercept of l_2.

Thus, l_2 has equation $y = 4x + b$. Since $Q(12, 0)$ lies on l_2, we have $0 = 4(12) + b$ which simplifies to $b = -48$.

Therefore, the equation of l_2 is $y = 4x - 48$ and the y-intercept of l_2 is -48.