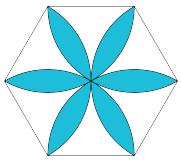


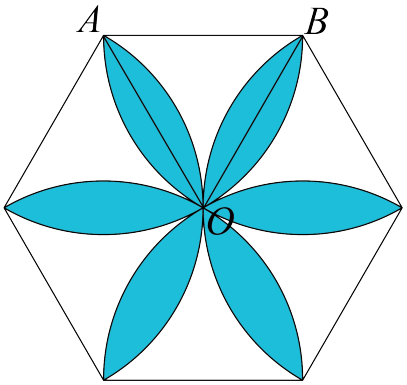
Problem of the Week
Problem D and Solution
A New Pi Plate

Problem

A pi-plate is in the shape of a regular hexagon with side length 8, and has an interesting pattern on it. From each vertex in the hexagon, a circular arc with centre at the vertex and radius 8 is drawn. This results in a flower-like pattern inside the hexagon, which is then shaded. Determine the total area of the shaded region.

Solution

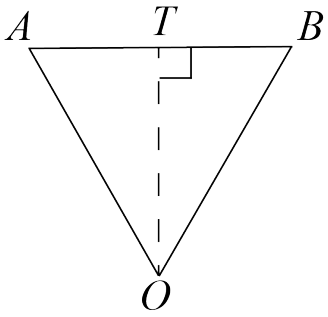
We label the centre of the hexagon O and consider two adjacent vertices, A and B. We draw in OA and OB.



We note that a regular hexagon is composed of six congruent equilateral triangles (the proof of this is left up to the student). Thus, $\triangle AOB$ is an equilateral triangle with side length 8.

To determine the total shaded area, we can consider the area of one-half of each 'petal' of the flower, and then multiply this area by 12, since there are 12 half-petals in the diagram. To determine the area of the region between the arc through B and O and the line segment OB, we can calculate the area of the sector AOB, with centre A and radius AB, and subtract the area of $\triangle AOB$.

We first determine the area of $\triangle AOB$. Construct altitude OT. Since $\triangle AOB$ is equilateral, it follows that OT bisects AB.



Since $AB = 8$, it follows that $AT = 4$. By the Pythagorean Theorem in $\triangle ATO$, $AO^2 = AT^2 + OT^2$. Therefore, $8^2 = 4^2 + OT^2$, and $OT^2 = 64 - 16 = 48$ follows. Since $OT > 0$, we have $OT = \sqrt{48}$.

Therefore, the area of $\triangle AOB$ is $\dfrac{(AB) \times (OT)}{2} = \dfrac{8 \times \sqrt{48}}{2} = 4\sqrt{48} = 16\sqrt{3}$.

We now determine the area of the sector AOB. Since $\angle OAB = 60°$, the area of sector AOB is $\frac{1}{6}$ of the area of a circle with radius 8. Thus, the area of the sector is equal to $\frac{1}{6}\pi(AB)^2 = \frac{1}{6}\pi(8)^2 = \frac{32}{3}\pi$.

Therefore, the area of the shaded region between the arc through B and O and the line segment OB is $\left(\frac{32}{3}\pi - 16\sqrt{3}\right)$ units^2.

The original pi-plate has 12 of these regions, so the entire shaded area is $12 \times (\frac{32}{3}\pi - 16\sqrt{3}) = 128\pi - 192\sqrt{3} \approx 69.57$ units^2.