Problem of the Week

Problems and Solutions 2025-2026

Problem C

Grade 7/8

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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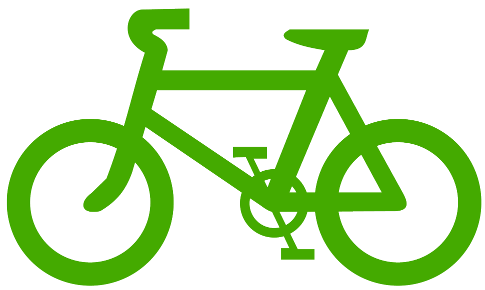
Problem C

Bicycle Trip

Cy Kler has mapped out a 560 km bike route that he wants to complete in seven days. Each day he wants to ride 15 km more than the day before.

If Cy is able to follow his plan, then how many kilometres will he have to ride on the seventh day of the trip?



Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Bicycle Trip

Problem

Cy Kler has mapped out a 560 km bike route that he wants to complete in seven days. Each day he wants to ride 15 km more than the day before.

If Cy is able to follow his plan, then how many kilometres will he have to ride on the seventh day of the trip?

Solution

Solution 1

We consider how much extra Cy Kler rides on each day after the first.

On Day 2, he rides 15 km extra.
On Day 3, he rides $15 + 15 = 30$ km extra.
On Day 4, he rides $15 + 15 + 15 = 45$ km extra.
On Day 5, he rides $15 + 15 + 15 + 15 = 60$ km extra.
On Day 6, he rides $15 + 15 + 15 + 15 + 15 = 75$ km extra.
On Day 7, he rides $15 + 15 + 15 + 15 + 15 + 15 = 90$ km extra.

Thus, the total distance is made up of seven day trips, each day beginning with the same distance as Day 1 plus $15 + 30 + 45 + 60 + 75 + 90 = 315$ km extra.

So, seven days of riding the same distance as Day 1 would total $560 - 315 = 245$ km. Thus, on Day 1, Cy Kler will ride $245 \div 7 = 35$ km.

On Day 7, Cy Kler will ride the Day 1 distance plus 90 km. Thus, the distance that he will ride on the seventh day is $35 + 90 = 125$ km.

Solutions 2 and 3 present more algebraic approaches to this problem.

Solution 2

Let x be the distance Cy Kler will ride on Day 1. Then he will ride $x + 15$, $x + 30$, $x + 45$, $x + 60$, $x + 75$, and $x + 90$ km on Day 2 through Day 7, respectively. Then,

$$\begin{aligned}
x + (x + 15) + (x + 30) + (x + 45) + (x + 60) + (x + 75) + (x + 90) &= 560\\
7x + 15 + 30 + 45 + 60 + 75 + 90 &= 560\\
7x + 315 &= 560\\
7x + 315 - 315 &= 560 - 315\\
7x &= 245\\
\frac{7x}{7} &= \frac{245}{7}\\
x &= 35
\end{aligned}$$

Thus, Cy Kler will ride 35 km on Day 1. Thus, the distance that he will ride on the seventh day is $x + 90 = 35 + 90 = 125$ km.

Solution 3

Let m be the distance Cy will ride on Day 4, the middle day. On Day 5 he would ride $(m + 15)$ km, on Day 6 he would ride $m + 15 + 15 = (m + 30)$ km, and on Day 7 he would ride $m + 30 + 15 = (m + 45)$ km. Working backwards from Day 4, we reduce the distance he rides by 15 km. On Day 3 he would ride $(m - 15)$ km, on Day 2 he would ride $m - 15 - 15 = (m - 30)$ km, and on Day 1 he would ride $m - 30 - 15 = (m - 45)$ km. Then,

$$\begin{aligned}
m + (m + 15) + (m + 30) + (m + 45) + (m - 15) + (m - 30) + (m - 45) &= 560\\
7m + 15 + 30 + 45 - 15 - 30 - 45 &= 560\\
7m + 15 - 15 + 30 - 30 + 45 - 45 &= 560\\
7m &= 560\\
\frac{7m}{7} &= \frac{560}{7}\\
m &= 80
\end{aligned}$$

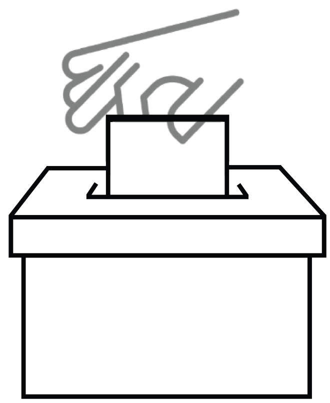
Thus, Cy Kler will ride 80 km on Day 4. Thus, the distance that he will ride on the seventh day is $m + 45 = 80 + 45 = 125$ km.

A solution like Solution 3 works well when there are an odd number of terms in a sequence that increases (or decreases) by a constant amount.

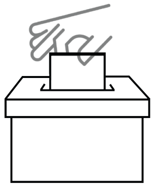
Problem of the Week
Problem C
Your Vote Counts

In an election for a representative on school council, there were only two candidates.

Sophia received 60% of the total votes and Mireille received all the rest. If Sophia won by 28 votes, then how many people voted?



Themes: Algebra, Number Sense



Problem of the Week
Problem C and Solution
Your Vote Counts

Problem

In an election for a representative on school council, there were only two candidates.

Sophia received 60% of the total votes and Mireille received all the rest. If Sophia won by 28 votes, then how many people voted?

Solution

Solution 1

If Sophia received 60% of the votes, then Mireille received 40% of the total number of votes. The difference between them, 20%, represents 28 votes.

We are interested in determining 100% of the votes, that is, the total number of votes cast. Since we know that 20% of the total votes cast was 28 votes and $5 \times 20 = 100$, then a total of 5×28 or 140 people voted.

Solution 2

The second solution uses an algebraic approach.

Let n represent the total number of votes cast.

Since Sophia received 60% of the total votes, she received $0.6 \times n$ or $0.6n$ votes.

Since Mireille received all of the remaining votes, she received $0.4 \times n$ or $0.4n$ votes.

The difference between the number of votes received by Sophia and the number of votes received by Mireille was 28. So,

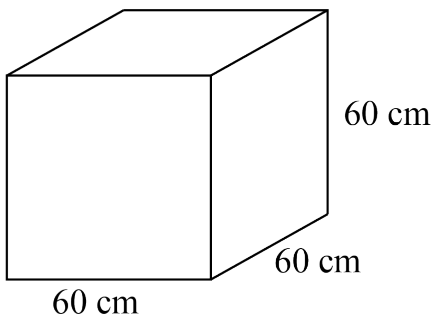
$$\begin{aligned} 0.6n - 0.4n &= 28 \\ 0.2n &= 28 \\ \frac{0.2n}{0.2} &= \frac{28}{0.2} \\ n &= 140 \end{aligned}$$

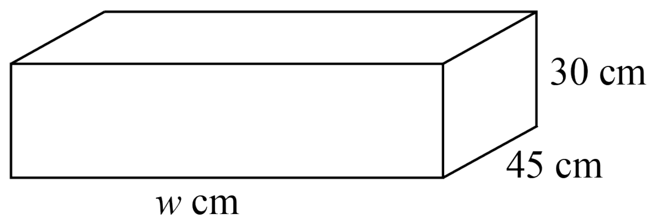
Therefore, 140 people voted.

Problem of the Week
Problem C
Hidden Space

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?





Themes: Algebra, Geometry & Measurement

Problem of the Week

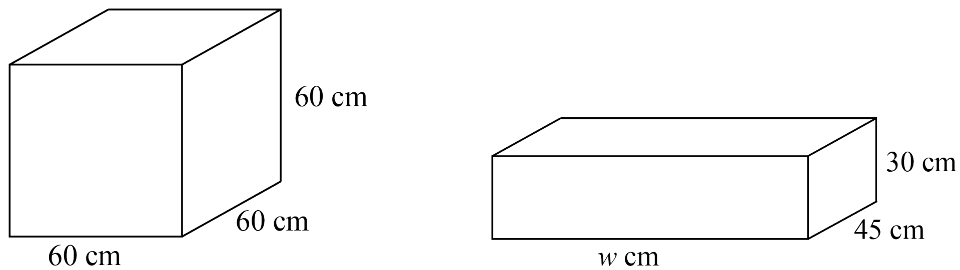
Problem C and Solution

Hidden Space

Problem

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?



Solution

To determine the surface area of a cube, we determine the area of one side and multiply by 6. Therefore, the total surface area of the cube-shaped box is $6 \times 60 \times 60 = 21\,600 \text{ cm}^2$.

To determine the total surface area of the rectangular prism, we determine the sum of the areas of the six sides. The front side and the back side have equal areas. The top and the bottom have equal areas. Each of the remaining two sides have equal area. Therefore, the total surface area of the rectangular prism box is

$$\begin{aligned} 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of side} &= 2 \times 30 \times w + 2 \times 45 \times w + 2 \times 30 \times 45 \\ &= 60w + 90w + 2700 \\ &= 150w + 2700 \end{aligned}$$

But the total surface area of the cube-shaped box is the same as the total surface area of the rectangular prism. Therefore,

$$\begin{aligned} 21\,600 &= 150w + 2700 \\ 21\,600 - 2700 &= 150w + 2700 - 2700 \\ 18\,900 &= 150w \\ \frac{18\,900}{150} &= \frac{150w}{150} \\ 126 &= w \end{aligned}$$

Therefore, the width of the rectangular prism is 126 cm.

To determine the volume of a cube, we "cube" the edge length. So, the volume of the cube is $60 \times 60 \times 60 = 60^3 = 216\,000 \text{ cm}^3$.

To determine the volume of the rectangular prism, we multiply the three different edge lengths. So, the volume of the rectangular prism is $126 \times 45 \times 30 = 170\,100 \text{ cm}^3$.

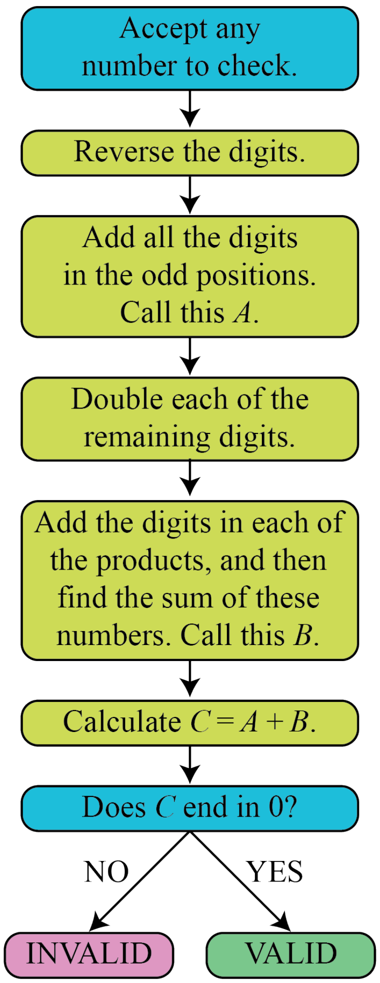
Therefore, the student who is building the cube has the box with the greater volume. The volume of the cube is greater by $216\,000 - 170\,100 = 45\,900 \text{ cm}^3$.

Problem of the Week

Problem C

Check Please 1

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$

$$\begin{aligned} A &= 2+7+3 \\ &= 12 \end{aligned}$$

- Double remaining digits:

$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$

- Calculate B:

$$\begin{aligned} B &= (1+8)+(1+0)+2 \\ &= 9+1+2 \\ &= 12 \end{aligned}$$

- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7+9+5+1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1+6)+(1+4)+6 \\ &= 7+5+6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number $13X9\ Y659\ 784$ is a valid card number when verified by the Luhn Algorithm, where X and Y are each single digits such that $X \leq Y$.

Determine all possible values of X and Y.

Themes: Algebra, Number Sense

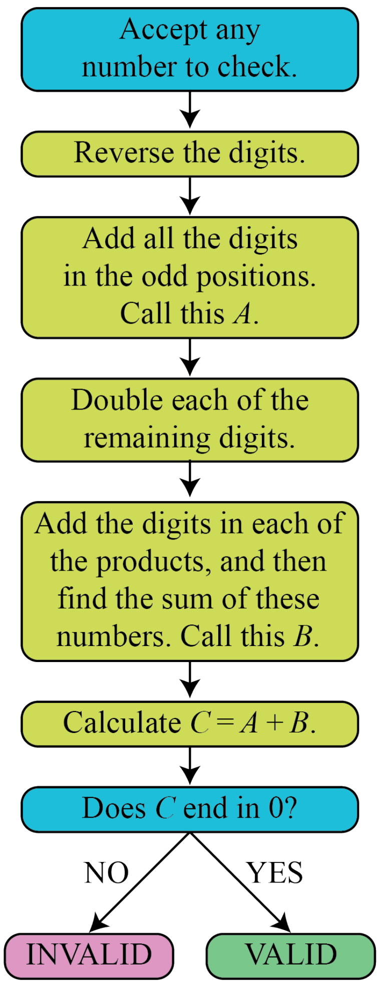
Problem of the Week

Problem C and Solution

Check Please 1

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



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- Calculate B:

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- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number $13X9\ Y659\ 784$ is a valid card number when verified by the Luhn Algorithm, where X and Y are each single digits such that $X \leq Y$.

Determine all possible values of X and Y.

Solution

When the digits of the card number are reversed the resulting number is $487\ 956Y\ 9X31$. The sum of the digits in the odd positions is $A = 4 + 7 + 5 + Y + X + 1 = 17 + X + Y$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$8 \times 2 = 16;\ 9 \times 2 = 18;\ 6 \times 2 = 12;\ 9 \times 2 = 18;\ 3 \times 2 = 6$$

When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + (1 + 8) + (1 + 2) + (1 + 8) + 6 = 7 + 9 + 3 + 9 + 6 = 34$$

Since $C = A + B$, we have $C = 17 + X + Y + 34 = 51 + X + Y$.

For the card to be valid, the units digit of C must be zero. Since X and Y are digits, the minimum value of $X + Y$ is 0 and the maximum is $9 + 9 = 18$. So C must be between 51 and $51 + 18 = 69$, inclusive. The only value in this range that has a units digit of 0 is 60. Thus, $C = 60$.

Since $C = 51 + X + Y = 60$, it follows that $X + Y = 9$. Since $X \leq Y$, the only possibilities for X and Y are shown in the following table:

X	Y
0	9
1	8
2	7
3	6
4	5

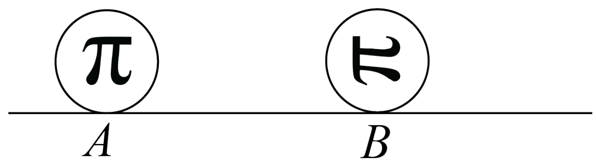
The corresponding card numbers are 1309 9659 784, 1319 8659 784, 1329 7659 784, 1339 6659 784, and 1349 5659 784, which are indeed verified by the Luhn Algorithm.

Problem of the Week

Problem C

Pi Day is Rolling Along

A coin has the image of π on one side and is being rolled along a horizontal surface. It starts at position A, with the π symbol oriented correctly. The coin then rolls until it reaches position B, which is 9 cm to the right of A. If the image of π is now rotated 90° counterclockwise for the first time, then what is the radius of the coin? Round your answer to 1 decimal. What orientation will the image of π be in after it has rolled a total of 174 cm?



Themes: Algebra, Geometry & Measurement

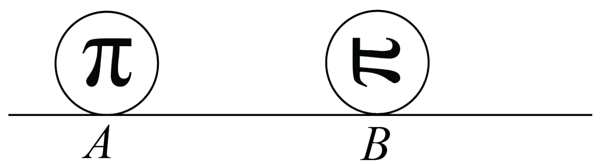
Problem of the Week

Problem C and Solution

Pi Day is Rolling Along

Problem

A coin has the image of π on one side and is being rolled along a horizontal surface. It starts at position A, with the π symbol oriented correctly. The coin then rolls until it reaches position B, which is 9 cm to the right of A. If the image of π is now rotated 90° counterclockwise for the first time, then what is the radius of the coin? Round your answer to 1 decimal. What orientation will the image of π be in after it has rolled a total of 174 cm?



Solution

At position B, the coin has rotated 90° counterclockwise for the first time. Thus, it must have gone through $\frac{3}{4}$ of a full rotation. Since the distance from A to B is 9 cm and the coin has gone through $\frac{3}{4}$ of a full rotation, a full rotation would move the coin 12 cm along the line from position A.

A full rotation is also completed by travelling one measure of the circumference of the coin.

Therefore, the circumference of the coin is equal to 12 cm.

We also know that $C = 2 \times \pi \times r$, where C is the circumference of a circle and r is its radius.

Thus, $12 = 2 \times \pi \times r$. Solving, we have $r = \dfrac{12}{2\pi} = \dfrac{6}{\pi} \approx 1.9$ cm.

Since the coin completes a full rotation every 12 cm, after it coin has rolled 174 cm along the line, it would have completed $\frac{174}{12} = 14.5$ revolutions.

This means the coin has completed 14 full revolutions and one-half of a revolution. Thus, the image of π will be upside down.

Problem of the Week
Problem C
Product Puzzle

Three cards are on a table. Each card has a letter on one side and a positive number on the other side. One card has an R on it, one card has a G on it, and one card has a B on it. The number side of each card is face down on the table. The following facts are known about the three concealed numbers:

(i) The product of the number on the card with an R and the number on the card with a G is equal to the number on the card with a B.

(ii) The product of the number on the card with a G and the number on the card with a B is 180.

(iii) Five times the number on the card with a B is equal to the number on the card with a G.

Determine the product of the numbers on the three cards.

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Product Puzzle

Problem

Three cards are on a table. Each card has a letter on one side and a positive number on the other side. One card has an R on it, one card has a G on it, and one card has a B on it. The number side of each card is face down on the table. The following facts are known about the three concealed numbers:

(i) The product of the number on the card with an R and the number on the card with a G is equal to the number on the card with a B.

(ii) The product of the number on the card with a G and the number on the card with a B is 180.

(iii) Five times the number on the card with a B is equal to the number on the card with a G.

Determine the product of the numbers on the three cards.

Solution
Solution 1

Let the three numbers be represented by r, g, and b.

Since the product of the number on the card with an R and the number on the card with a G is equal to the number on the card with a B, $r \times g = b$. We are looking for $r \times g \times b = (r \times g) \times b = (b) \times b = b^2$. So when we find b^2 we have found the required product $r \times g \times b$.

We are also given that $g \times b = 180$ and $g = 5 \times b$. Substituting $g = 5 \times b$ into $g \times b = 180$, we have $(5 \times b) \times b = 180$ or $5 \times b^2 = 180$. Dividing by 5, we obtain $b^2 = 36$. This is exactly what we are looking for since $r \times g \times b = b^2$.

Therefore, the product of the three numbers is 36.

For those who would like to know the values of the three numbers, we can continue to solve. We know $b^2 = 36$, so $b = 6$, since b is a positive number. Therefore, $g = 5 \times b = 5 \times 6 = 30$. And finally, $r \times g = b$ so $r \times (30) = 6$. Dividing by 30, we get $r = \frac{6}{30} = \frac{1}{5}$.

We can verify the product $r \times g \times b = (\frac{1}{5}) \times (30) \times (6) = 6 \times 6 = 36$.

Solution 2

In this solution we try to determine the numbers by working with the factors of 180.

The product of the number on the card with a G and the number on the card with a B is 180, and the number on the card with a G is five times the number on the card with a B. The number 180 can be written as $2 \times 2 \times 3 \times 3 \times 5$. If G and B are integers, then G and B must share the same factors, along with G having one more factor, that being 5. By playing with the factors of 180, we find a possibility is that the number on the card with a G is $5 \times 2 \times 3$ and the number on the card with a B is 2×3. That is, the number on the card with a G could be 30 and the number on the card with a B could be 6.

Now using the fact that the number on the card with an R multiplied by the number on the card with a G is equal to the number on the card with a B, we determine that some number multiplied by 30 is equal to 6. It follows that the number on the card with an R would be $6 \div 30 = \frac{1}{5}$.

The product of the three numbers is $\frac{1}{5} \times 30 \times 6 = 6 \times 6 = 36$.

This solution works because the number on the card with a G and the number on the card with a B happen to be integers.

Problem of the Week
Problem C
Soccer Practice

Chris and Pat are practicing soccer. Standing 1 m apart, Pat first kicks the ball to Chris and then Chris kicks the ball back to Pat. Next, standing 2 m apart, Pat kicks the ball to Chris and Chris kicks it back to Pat. After each pair of kicks, Chris moves 1 m farther away from Pat. If they stop playing after the 29^{th} kick, how far apart are they standing and how far did the ball travel in total?

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Soccer Practice

Problem

Chris and Pat are practicing soccer. Standing 1 m apart, Pat first kicks the ball to Chris and then Chris kicks the ball back to Pat. Next, standing 2 m apart, Pat kicks the ball to Chris and Chris kicks it back to Pat. After each pair of kicks, Chris moves 1 m farther away from Pat. If they stop playing after the 29$^{\text{th}}$ kick, how far apart are they standing and how far did the ball travel in total?

Solution

At each distance, two kicks are made: the 1$^{\text{st}}$ and 2$^{\text{nd}}$ kicks are made when the pair is 1 m apart, the 3$^{\text{rd}}$ and 4$^{\text{th}}$ kicks are made when they are 2 m apart, and so on, with the 27$^{\text{th}}$ and 28$^{\text{th}}$ kicks being made when they are 14 m apart.
Therefore, the 29$^{\text{th}}$ kick is the first kick made when they are 15 m apart. At each distance, the first kick is made by Pat to Chris, so Pat does the 29$^{\text{th}}$ kick at a distance of 15 m.

The total distance travelled by the ball is equal to

$$\begin{aligned}
&1+1+2+2+3+3+\cdots+14+14+15\\
&=(1+2+3+\cdots+14)+(1+2+3+\cdots+14+15)\\
&=\frac{14\times 15}{2}+\frac{15\times 16}{2}\\
&=105+120\\
&=225
\end{aligned}$$

Therefore, the ball travelled a total distance of 225 m.

NOTE:
Did you know that the sum of the first n positive integers is equal to $\frac{n\times(n+1)}{2}$?
That is,

$$1+2+3+\cdots+n=\frac{n\times(n+1)}{2}$$

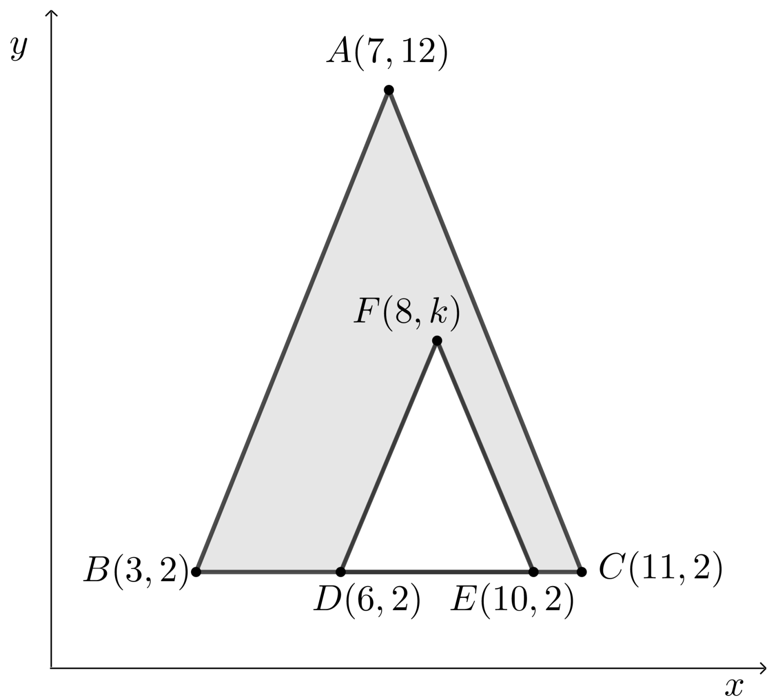
For example, $1+2+3+4+5=15$, and $\dfrac{5\times 6}{2}=15$.

Also, $1+2+3+4+5+6+7+8=36$, and $\dfrac{8\times 9}{2}=36$.

Problem of the Week
Problem C
Tangled Triangles

Points $A(7,12)$, $B(3,2)$, $C(11,2)$, $D(6,2)$ and $E(10,2)$ are plotted on the Cartesian plane. The point $F(8,k)$ lies inside $\triangle ABC$ so that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2.

Determine the value of k, the y-coordinate of F.



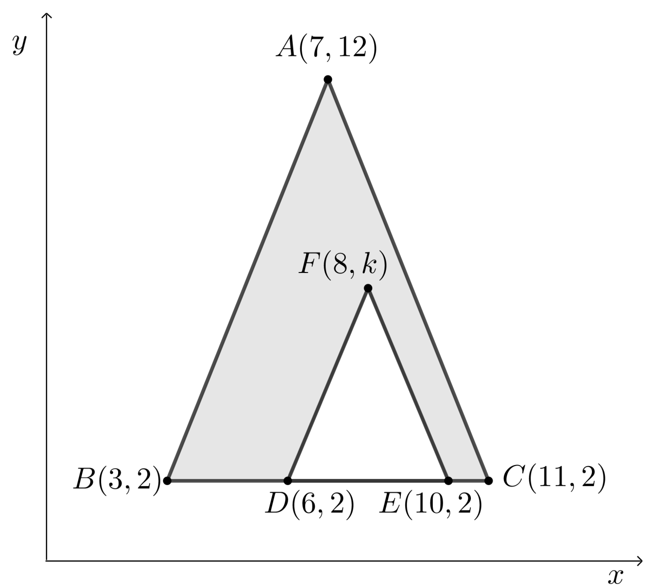
Themes: Algebra, Geometry & Measurement

Problem of the Week
Problem C and Solution
Tangled Triangles

Problem

Points $A(7, 12)$, $B(3, 2)$, $C(11, 2)$, $D(6, 2)$ and $E(10, 2)$ are plotted on the Cartesian plane. The point $F(8, k)$ lies inside $\triangle ABC$ so that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2.

Determine the value of k, the y-coordinate of F.



Solution

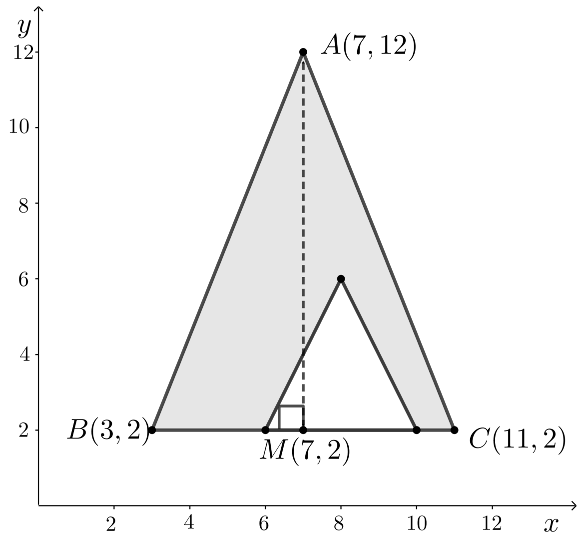
We will calculate the area of $\triangle ABC$, calculate the area of $\triangle DEF$, and then use the given information that the difference between these areas is equal to 32 units^2.

We will use the fact that the distance between two points that have the same x-coordinate is the positive difference between their y-coordinates. We will also use the fact that the distance between two points that have the same y-coordinate is the positive difference between their x-coordinates.

Since B and C both have y-coordinate 2, BC is a horizontal line. Thus, $BC = 11 - 3 = 8$.

In $\triangle ABC$, drop a perpendicular from vertex A to M on BC. Since BC is horizontal, then AM is vertical. Since every point on a vertical line has the same x-coordinate, M has x-coordinate 7. Similarly, since M is on the horizontal line through $B(3, 2)$ and $C(11, 2)$, M has y-coordinate 2. Therefore, $AM = 12 - 2 = 10$.

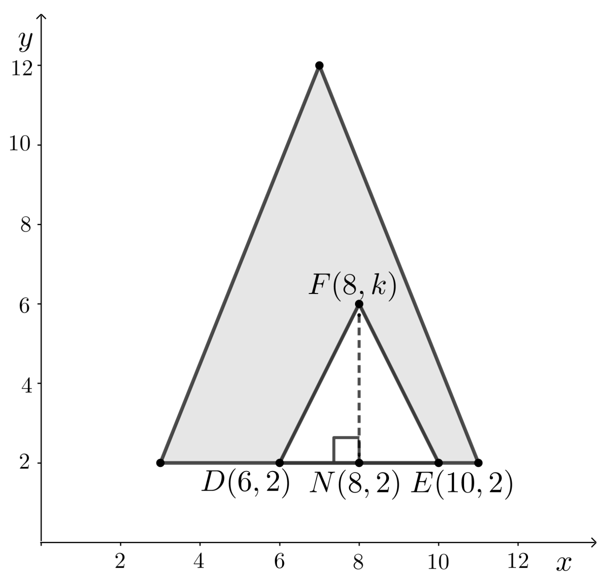
Thus, the area of $\triangle ABC$ is equal to $\dfrac{BC \times AM}{2} = \dfrac{8 \times 10}{2} = 40$.



Since D and E both have y-coordinate 2, DE is a horizontal line. Thus, $DE = 10 - 6 = 4$.

In $\triangle DEF$, drop a perpendicular from vertex F to N on DE. Since DE is horizontal, then FN is vertical. Since every point on a vertical line has the same x-coordinate, N has x-coordinate 8. Similarly, since N is on the horizontal line through $D(6,2)$ and $E(10,2)$, N has y-coordinate 2. Therefore, $FN = k - 2$.

Thus, the area of $\triangle DEF$ is equal to $\dfrac{DE \times FN}{2} = \dfrac{4 \times (k-2)}{2} = 2(k-2)$.



We now use the given information that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2 to obtain the equation $40 - 2(k-2) = 32$.

Subtracting 32 from each side, we have $8 - 2(k-2) = 0$.

Adding $2(k-2)$ to each side, we have $8 = 2(k-2)$, which simplifies to $4 = k - 2$, or $k = 6$.

Therefore, the value of k, the y-coordinate of point F, is 6.

Problem of the Week
Problem C
Just a Little Output

Harlow has written a program. After she inputs a number into the program the number is multiplied by 2, then the product is divided by 3, then 5 is subtracted from the quotient. The final result is then printed.

For example, if Harlow inputs the number 60 into the program then following the steps outlined, the number 60 is first multiplied by 2 to obtain $60 \times 2 = 120$. Then the product is divided by 3 to obtain $120 \div 3 = 40$. Finally, 5 is subtracted from the quotient to obtain $40 - 5 = 35$. The number 35 is then printed.

If Harlow inputs only positive integers into the program, determine the smallest possible positive output number that can be printed.

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Just a Little Output

Problem

Harlow has written a program. After she inputs a number into the program the number is multiplied by 2, then the product is divided by 3, then 5 is subtracted from the quotient. The final result is then printed.

For example, if Harlow inputs the number 60 into the program then following the steps outlined, the number 60 is first multiplied by 2 to obtain $60 \times 2 = 120$. Then the product is divided by 3 to obtain $120 \div 3 = 40$. Finally, 5 is subtracted from the quotient to obtain $40 - 5 = 35$. The number 35 is then printed.

If Harlow inputs only positive integers into the program, determine the smallest possible positive output number that can be printed.

Solution

We can start by trying different input numbers, starting with 1 and increasing by 1 each time, to see if the output numbers form a pattern. The results for inputs from 1 to 5 are in the following table.

Input	After Multiplying by 2	After Dividing by 3	After Subtracting 5
1	$1 \times 2 = 2$	$2 \div 3 = \frac{2}{3}$	$\frac{2}{3} - \frac{15}{3} = -\frac{13}{3}$
2	$2 \times 2 = 4$	$4 \div 3 = \frac{4}{3}$	$\frac{4}{3} - \frac{15}{3} = -\frac{11}{3}$
3	$3 \times 2 = 6$	$6 \div 3 = 2$	$2 - 5 = -3$
4	$4 \times 2 = 8$	$8 \div 3 = \frac{8}{3}$	$\frac{8}{3} - \frac{15}{3} = -\frac{7}{3}$
5	$5 \times 2 = 10$	$10 \div 3 = \frac{10}{3}$	$\frac{10}{3} - \frac{15}{3} = -\frac{5}{3}$

From this table we can see that when the input numbers are positive integers starting at 1 and increasing by 1 each time, the output numbers form a linear sequence that increases by $\frac{2}{3}$ each time. We now proceed with two different solutions.

Solution 1

In this solution we continue the table until we get the first positive output value.

Input	After Multiplying by 2	After Dividing by 3	After Subtracting 5
6	$6 \times 2 = 12$	$12 \div 3 = 4$	$4 - 5 = -1$
7	$7 \times 2 = 14$	$14 \div 3 = \frac{14}{3}$	$\frac{14}{3} - \frac{15}{3} = -\frac{1}{3}$
8	$8 \times 2 = 16$	$16 \div 3 = \frac{16}{3}$	$\frac{16}{3} - \frac{15}{3} = \frac{1}{3}$

From the table, the first positive output number is $\frac{1}{3}$, and happens when the input number is 8. Since the output numbers form an increasing sequence, we know that $\frac{1}{3}$ is also the smallest possible positive output number.

Note that this solution worked well here because the input number for the smallest possible positive output number was fairly small. If the input number had been much larger than 8 then this wouldn't have been an efficient solution.

Solution 2

This solution uses algebra to solve the problem. Let n be the input number. Following the steps in the program, we first multiply by 2 to obtain $2 \times n$. We then divide by 3 to obtain $\frac{2\times n}{3}$. Finally, we subtract 5 from this quotient to obtain $\frac{2\times n}{3} - 5$. Thus, when the input is n, the output is $\frac{2\times n}{3} - 5$. Since we want the output number to be positive, then

$$\begin{aligned}
\frac{2 \times n}{3} - 5 &> 0 \\
\frac{2 \times n}{3} &> 5 \\
2 \times n &> 5 \times 3 \\
2 \times n &> 15 \\
n &> 15 \div 2 \\
n &> 7.5
\end{aligned}$$

Thus, the input number must be greater than 7.5. Since the input number must be an integer, it follows that the smallest input number with a positive output number is 8. Since the output numbers form an increasing sequence, an input of 8 will give the smallest possible positive output number. When the input number is 8, the output number is:

$$\frac{2 \times 8}{3} - 5 = \frac{16}{3} - 5 = \frac{16}{3} - \frac{15}{3} = \frac{1}{3}$$

Therefore, if the input number is a positive integer, then the smallest possible positive output number is $\frac{1}{3}$.

Computational Thinking (C)

Take me to the cover

Problem of the Week

Problem C

Jumbled Numbers 1

Imani wrote a program that switches digits in numbers. Users enter a 4-digit number and then use the following buttons to switch digits in the number:

Button	Function
red	switches the thousands and tens digits
blue	switches the hundreds and units (ones) digits
yellow	switches the thousands and units (ones) digits

Imani enters the last four digits of her phone number, then presses the following sequence of buttons, in order from left to right.

red yellow blue red yellow

The number now shown is 2148.

Then, Imani restarts the program, enters the last four digits of her phone number again, and presses the following sequence of buttons, in order from left to right.

blue yellow red blue

What number is now shown?

Theme: Computational Thinking

Problem of the Week
Problem C and Solution
Jumbled Numbers 1

Problem

Imani wrote a program that switches digits in numbers. Users enter a 4-digit number and then use the following buttons to switch digits in the number:

Button	Function
red	switches the thousands and tens digits
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Imani enters the last four digits of her phone number, then presses the following sequence of buttons, in order from left to right.

red yellow blue red yellow

The number now shown is 2148.

Then, Imani restarts the program, enters the last four digits of her phone number again, and presses the following sequence of buttons, in order from left to right.

blue yellow red blue

What number is now shown?

Solution

First we need to determine the last four digits of Imani's phone number, as this is the number she entered into the program. We can determine this by working backwards starting with 2148, which is the number shown after all the buttons were pressed. We will then go through the buttons one by one, from last to first.

- To obtain 2148, the number must have been 8142 before pressing yellow.
- Then, the number must have been 4182 before pressing red.
- Then, the number must have been 4281 before pressing blue.
- Then, the number must have been 1284 before pressing yellow.
- Then, the number must have been 8214 before pressing red.

Thus, the last four digits of Imani's phone number are 8214.

Now we can determine the number shown after Imani entered 8214 and pressed the second sequence of buttons.

- The number will be 8412 after pressing blue.
- Then, the number will be 2418 after pressing yellow.
- Then, the number will be 1428 after pressing red.
- Then, the number will be 1824 after pressing blue.

Thus, the number 1824 is now shown.

Problem of the Week

Problem C

The First Escape

Ingrid, Jayant, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Grandma's Attic, In Plain Sight, or The Vault. The three groups all started at the same time, but finished at different times. Use the clues below to match each group with the name of their escape room, and the order in which they finished.

(1) Ingrid's group finished before Radko's group.

(2) The group that completed Grandma's Attic was not Radko's group.

(3) The Vault was completed before In Plain Sight.

(4) Radko's group finished after Jayant's group.

(5) Ingrid's group did not complete The Vault.

(6) The group that finished first did not complete Grandma's Attic.

Theme: Computational Thinking

Problem of the Week
Problem C and Solution
The First Escape

Problem

Ingrid, Jayant, and Radko each brought a group of friends to POTW Escape. Each group completed a different escape room: Grandma's Attic, In Plain Sight, or The Vault. The three groups all started at the same time, but finished at different times. Use the clues below to match each group with the name of their escape room, and the order in which they finished.

(1) Ingrid's group finished before Radko's group.

(2) The group that completed Grandma's Attic was not Radko's group.

(3) The Vault was completed before In Plain Sight.

(4) Radko's group finished after Jayant's group.

(5) Ingrid's group did not complete The Vault.

(6) The group that finished first did not complete Grandma's Attic.

Solution

From clues (1) and (4), since Ingrid's group finished before Radko's group and Radko's group finished after Jayant's group, we can conclude that Radko's group finished last.

Then, from clue (2) we can conclude that Grandma's Attic was completed either first or second. However, clue (6) states that the group that finished first did not complete Grandma's Attic. So Grandma's Attic must have been completed second. We summarize what we know so far in the following table.

	Last to Finish	**Second to Finish**	**First to Finish**
Group Leader	Radko		
Room Name		Grandma's Attic	

Then, from clue (3) we can conclude that The Vault was completed first and In Plain Sight was completed last. We summarize what we know so far in the following table.

	Last to Finish	**Second to Finish**	**First to Finish**
Group Leader	Radko		
Room Name	In Plain Sight	Grandma's Attic	The Vault

Finally, from clue (5) we can conclude that Ingrid's group completed Grandma's Attic, so then Jayant's group completed The Vault. The completed table is shown.

	Last to Finish	**Second to Finish**	**First to Finish**
Group Leader	Radko	Ingrid	Jayant
Room Name	In Plain Sight	Grandma's Attic	The Vault

Data Management (D)

Take me to the cover

Problem of the Week
Problem C
Sweet Treats

Nadia can confidently make 32 different desserts. Sixteen of the desserts contain gluten and twelve contain dairy. Seven of the desserts contain both gluten and dairy. How many of the desserts are both gluten free and dairy free?

Themes: Data Management, Number Sense

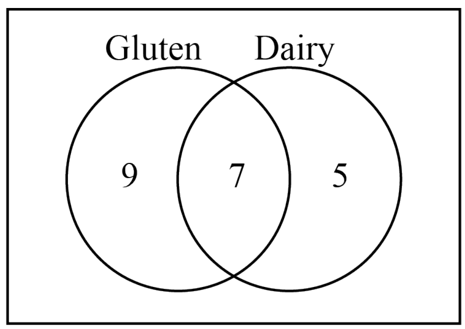
Problem of the Week
Problem C and Solution
Sweet Treats

Problem

Nadia can confidently make 32 different desserts. Sixteen of the desserts contain gluten and twelve contain dairy. Seven of the desserts contain both gluten and dairy. How many of the desserts are both gluten free and dairy free?

Solution

Since 7 desserts contain both gluten and dairy, and these are included in the 16 desserts that contain gluten, then $16 - 7 = 9$ desserts contain gluten but not dairy. Similarly, the 7 desserts that contain both gluten and dairy are included in the 12 desserts that contain dairy, so $12 - 7 = 5$ desserts contain dairy but not gluten. This is shown in the following Venn diagram.



The desserts can then be separated into four categories: desserts that contain gluten but not dairy, desserts that contain dairy but not gluten, desserts that contain both dairy and gluten, and desserts that are both gluten free and dairy free. If we add the number of desserts in each category, we will get the total number of desserts that Nadia can make, which is 32. Thus, the number of desserts that are both gluten free and dairy free is equal to the total number of desserts minus the number of desserts that contain gluten but not dairy minus the number of desserts that contain dairy but not gluten minus the number of desserts that contain both gluten and dairy.

Since $32 - 9 - 5 - 7 = 11$, it follows that the number of desserts that are both gluten free and dairy free is 11.

Problem of the Week
Problem C
Math Test Math 1

Dilip has had six tests in his math class so far. He knows the average (mean) of some of his marks.

- The average of his first and second test marks is 77%.
- The average of his third and fourth test marks is 75%.
- The average of his fifth and sixth test marks is 82%.

Determine the average of all six test marks for Dilip.

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Math Test Math 1

Problem

Dilip has had six tests in his math class so far. He knows the average (mean) of some of his marks.

- The average of his first and second test marks is 77%.
- The average of his third and fourth test marks is 75%.
- The average of his fifth and sixth test marks is 82%.

Determine the average of all six test marks for Dilip.

Solution

The average (mean) of all six test marks is equal to the sum of all six test marks divided by 6.

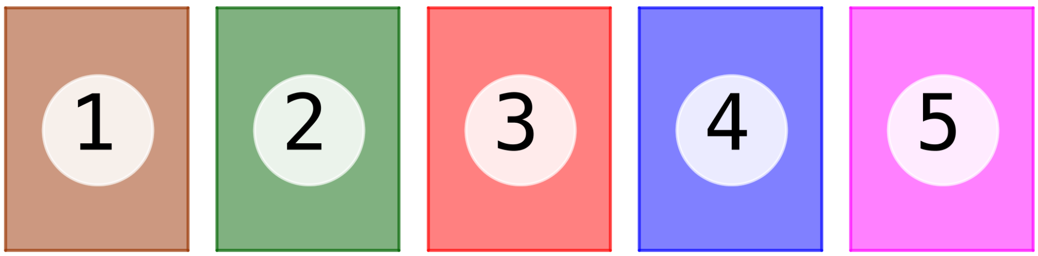
Since the average of Dilip's first and second test marks is 77, it follows that the sum of the first and second test marks is $2 \times 77 = 154$. Similarly, the sum of his third and fourth test marks is $2 \times 75 = 150$, and the sum of his fifth and sixth test marks is $2 \times 82 = 164$. It follows that the sum of all six test marks is $154 + 150 + 164 = 468$. Since $468 \div 6 = 78$, it follows that the average of all six test marks is 78%.

Problem of the Week

Problem C

Three Card Sum

Five cards are numbered with the integers 1 through 5. The cards are mixed up and placed face down on a table. A person randomly flips over three cards. What is the probability that the sum of the numbers on those three cards is odd?



Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Three Card Sum

Problem

Five cards are numbered with the integers 1 through 5. The cards are mixed up and placed face down on a table. A person randomly flips over three cards. What is the probability that the sum of the numbers on those three cards is odd?

Solution

In order to determine the probability, we must determine the number of ways to obtain a sum that is odd and divide it by the total number of possibilities for the three flipped cards. We will count the possibilities for the three flipped cards by systematically working through cases.

- Case 1: They flip 1 and 2 and one higher number. Then there are 3 possibilities for the higher number: 3, 4, and 5.
- Case 2: They flip 1 and 3 and one higher number. Then there are 2 possibilities for the higher number: 4 and 5.
- Case 3: They flip 1 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.
- Case 4: They flip 2 and 3 and one higher number. Then there are 2 possibilities for the higher number: 4 and 5.
- Case 5: They flip 2 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.
- Case 6: They flip 3 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.

Therefore, there are $3+2+1+2+1+1=10$ possible flips of three cards. How many of these have an odd sum? We could take each of the possibilities, determine the sum, and then count the number which produce an odd sum. We will present a different method.

The sum of three integers is odd in two instances: all three integers are odd or one integer is odd and the other two are even.

There is only one possibility with three odd integers. This is when 1, 3, and 5 are flipped. There are three possibilities with one odd integer and two even integers. They are when 1, 2, and 4 are flipped, or when 2, 3, and 4 are flipped, or when 2, 4, and 5 are flipped. Therefore, there are $1+3=4$ flips where the sum is odd.

Therefore, the probability of flipping three cards with an odd sum is $\frac{4}{10}=\frac{2}{5}$.

Extension: If nine cards numbered with the integers 1 through 9 are mixed up and placed face down on a table, then the probability of flipping over three cards that have an odd sum is $\frac{10}{21}$. Can you verify this?

Problem of the Week

Problem C

Favourite Number 1

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Isaac announced that his favourite number is a five-digit positive integer with a digit sum of 3. Radhika then wrote down a five-digit positive integer with a digit sum of 3. What is the probability that the number Radhika wrote was Isaac's favourite number?

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Favourite Number 1

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Isaac announced that his favourite number is a five-digit positive integer with a digit sum of 3. Radhika then wrote down a five-digit positive integer with a digit sum of 3. What is the probability that the number Radhika wrote was Isaac's favourite number?

Solution

We will first find the groups of five digits that add to 3. Then we will rearrange these digits to determine all five-digit positive integers whose digit sum is 3. Note that the first digit cannot be 0, because otherwise the integer would not be a five-digit integer. This is summarized in the following table.

The five digits	The possible five-digit integers	Number of possibilities
$3, 0, 0, 0, 0$	30 000	1
$1, 2, 0, 0, 0$	12 000, 21 000, 10 200, 20 100, 10 020, 20 010, 10 002, 20 001	8
$1, 1, 1, 0, 0$	11 100, 10 110, 11 010, 10 101, 11 001, 10 011	6

Therefore, the number of five-digit positive integers that have a digit sum of 3 is $1 + 8 + 6 = 15$. Since Radhika wrote down only one of these integers, the probability that this integer was Isaac's favourite number is $\frac{1}{15}$.

Note: It is a known fact that an integer is divisible by 3 exactly when its digit sum is divisible by 3. For example, 32 814 has a digit sum of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 3, then 32 814 is divisible by 3. On the other hand, 32 810 has a digit sum of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 3, then 32 810 is not divisible by 3.

As a consequence of this fact, Isaac's favourite number must be divisible by 3.

Problem of the Week
Problem C
Choir Sections

There are 30 students in the school choir at Pascal Middle School. Their conductor split them into two singing groups: altos and sopranos. In total there are 13 altos and 17 sopranos in the choir.

One day there were some students absent from choir practice. Their conductor noticed that more than half the students in the choir were present, and two-thirds of the students present were altos. How many students were absent from choir practice that day?

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Choir Sections

Problem

There are 30 students in the school choir at Pascal Middle School. Their conductor split them into two singing groups: altos and sopranos. In total there are 13 altos and 17 sopranos in the choir.

One day there were some students absent from choir practice. Their conductor noticed that more than half the students in the choir were present, and two-thirds of the students present were altos. How many students were absent from choir practice that day?

Solution

Solution 1

Since more than half the students in the choir were present that day, that means there were at least 16 students present. Since there were some students absent, that means there were at most 29 students present. Since two-thirds of the students present were altos, that tells us the number of students present must be a multiple of 3. Thus, the number of students present is a multiple of 3 between 16 and 29. The possibilities are 18, 21, 24, and 27.

If there were 18 students present, then $\frac{2}{3} \times 18 = 12$ of them were altos and $18 - 12 = 6$ of them were sopranos. This is a possibility since there are 13 altos and 17 sopranos in the choir.

If there were 21 students present, then $\frac{2}{3} \times 21 = 14$ of them were altos and $21 - 14 = 7$ of them were sopranos. However there are only 13 altos in the choir, so it's not possible that 14 of the students were altos. Thus, this is not a possibility.

If there were more than 21 students present, then the number of altos would be more than 14. Thus, 24 and 27 are also not possibilities for the number of students present.

Therefore, there were 18 students present that day, so $30 - 18 = 12$ students were absent.

Solution 2

This solution uses algebra to solve the problem. Note that the algebra used may be beyond what students have done so far in their math classes.

Let a be the number of altos absent and let s be the number of sopranos absent. Since some students were absent, but more than half the class was present, it follows that the total number of students absent is greater than 0 but less than 15. Thus, $a + s > 0$ and $a + s < 15$.

The number of altos present that day was $13 - a$ and the number of sopranos present was $17 - s$. Since two-thirds of the students present were altos, it follows that one-third of the students present were sopranos. So there were twice as many altos as sopranos. Thus,

$$\begin{aligned} 13 - a &= 2(17 - s) \\ 13 - a &= 34 - 2s \\ 0 &= 21 + a - 2s \\ a &= 2s - 21 \end{aligned}$$

Since a represents the number of altos absent, it follows that $a \geq 0$. So the smallest possible value for s is 11, since $2(10) - 21 = 20 - 21 = -1 < 0$ but $2(11) - 21 = 22 - 21 = 1$. Thus, if $s = 11$ then $a = 1$. In this case, the total number of students absent would be $11 + 1 = 12$.

If $s = 12$, then $a = 2(12) - 21 = 24 - 21 = 3$. Then the total number of students absent would be $12 + 3 = 15$. However this is too many, since the number of students absent must be less than 15. If s is any other number larger than 12 then there will be too many students absent. Thus $s = 11$, $a = 1$, and the total number of students absent that day was 12.

Geometry & Measurement (G)

Take me to the cover

Problem of the Week

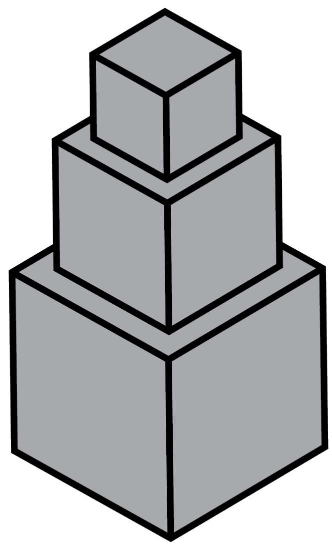
Problem C

Tower of Cubes

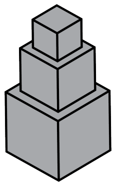
Imtiaz is an artist. One of the pieces of art that he created is a tower of three cubes. The bottom cube has a side length of 3 m, the middle cube has a side length of 2 m, and the top cube has a side length of 1 m. The top two cubes are each centred on the cube below.

Imtiaz wishes to paint the piece of art. Since the piece will be suspended in the air, the bottom will also be painted.

Determine the total surface area of the piece of art, including the bottom.



Theme: Geometry & Measurement



Problem of the Week
Problem C and Solution
Tower of Cubes

Problem

Imtiaz is an artist. One of the pieces of art that he created is a tower of three cubes. The bottom cube has a side length of 3 m, the middle cube has a side length of 2 m, and the top cube has a side length of 1 m. The top two cubes are each centred on the cube below.

Imtiaz wishes to paint the piece of art. Since the piece will be suspended in the air, the bottom will also be painted.

Determine the total surface area of the piece of art, including the bottom.

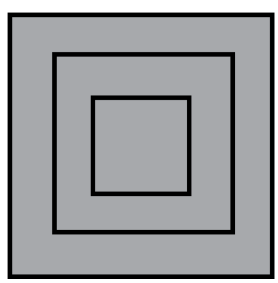
Solution

To determine the areas of the faces, we will use the formula for the area of a rectangle, Area = length × width.

Each cube has 4 exposed square sides, so the total area of all the side faces is

$$\begin{aligned} 4 \times (1 \times 1) + 4 \times (2 \times 2) + 4 \times (3 \times 3) &= 4 \times (1) + 4 \times (4) + 4 \times (9) \\ &= 4 + 16 + 36 \\ &= 56 \text{ m}^2 \end{aligned}$$

To determine the total area of the exposed tops of the cubes, we look down on the tower and see a image like the one below.



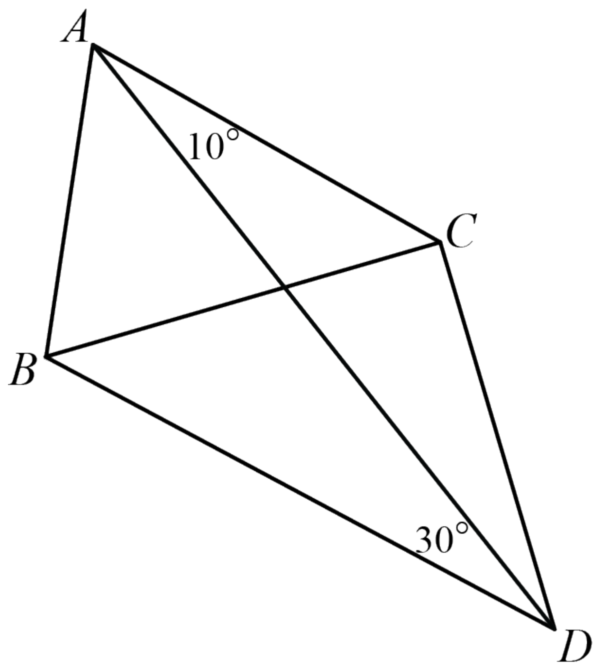
This exposed area is equal to the side area of one face of the bottom cube. Therefore, the total area of the exposed tops of the cubes is $3 \times 3 = 9$ m^2. The bottom face to be painted also has area $3 \times 3 = 9$ m^2.

Therefore, the total surface area is $56 + 9 + 9 = 74$ m^2.

Extension: Three cubes with side lengths x, y and z are stacked on top of each other in a similar manner to the original problem such that $0 < x < y < z$. Show that the total surface area of the tower, including the bottom, is $6z^2 + 4y^2 + 4x^2$.

Problem of the Week
Problem C
Subtle Symmetry

Quadrilateral $ACDB$ has $AC = BC = DC$, $\angle ADB = 30^\circ$, and $\angle CAD = 10^\circ$.



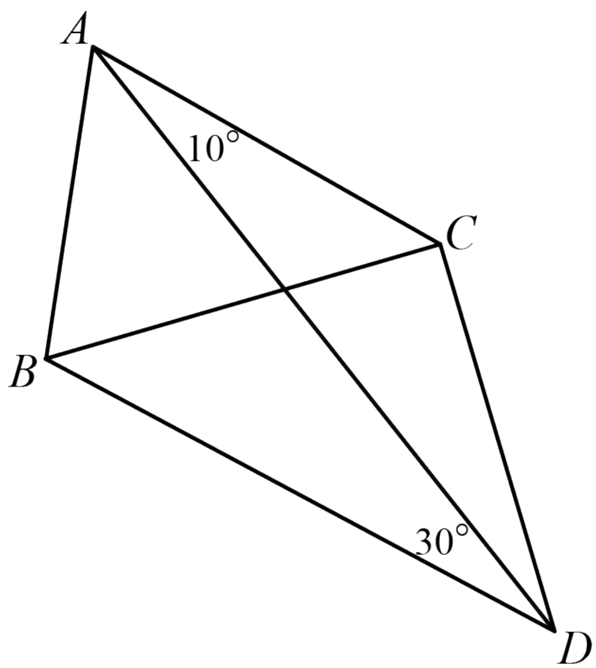
Determine the measure of $\angle ACB$.

Theme: Geometry & Measurement

Problem of the Week
Problem C and Solution
Subtle Symmetry

Problem

Quadrilateral $ACDB$ has $AC = BC = DC$, $\angle ADB = 30^\circ$, and $\angle CAD = 10^\circ$.



Determine the measure of $\angle ACB$.

Solution

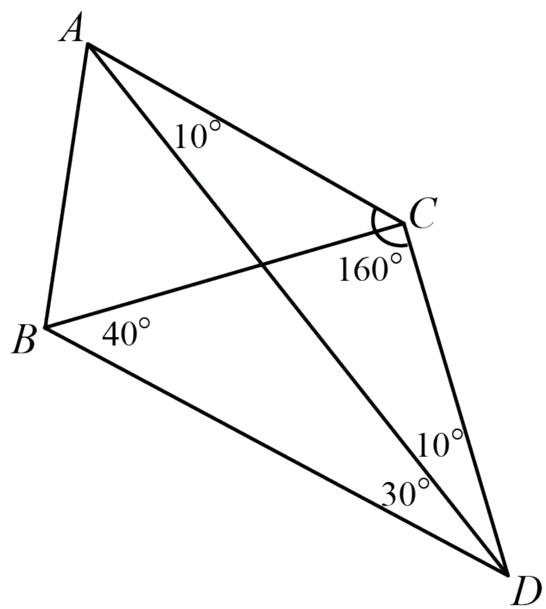
Since $AC = DC$, $\triangle ACD$ is isosceles. Therefore, $\angle CDA = \angle CAD = 10^\circ$.

Thus, $\angle CDB = \angle CDA + \angle ADB = 10^\circ + 30^\circ = 40^\circ$.

Since $BC = DC$, $\triangle BCD$ is isosceles. Therefore, $\angle CBD = \angle CDB = 40^\circ$.

Since the angles in a triangle sum to 180°, in $\triangle ACD$ we have

$$\begin{aligned} \angle CAD + \angle CDA + \angle ACD &= 180^\circ \\ 10^\circ + 10^\circ + \angle ACD &= 180^\circ \\ 20^\circ + \angle ACD &= 180^\circ \\ \angle ACD &= 160^\circ \end{aligned}$$



Since the angles in a triangle sum to 180°, in $\triangle BCD$ we have

$$\begin{aligned} \angle CDB + \angle CBD + \angle BCD &= 180^\circ \\ 40^\circ + 40^\circ + \angle BCD &= 180^\circ \\ 80^\circ + \angle BCD &= 180^\circ \\ \angle BCD &= 100^\circ \end{aligned}$$

Since $\angle ACD = \angle ACB + \angle BCD$, we have $160^\circ = \angle ACB + 100^\circ$. Therefore, $\angle ACB = 160^\circ - 100^\circ = 60^\circ$.

Extension:

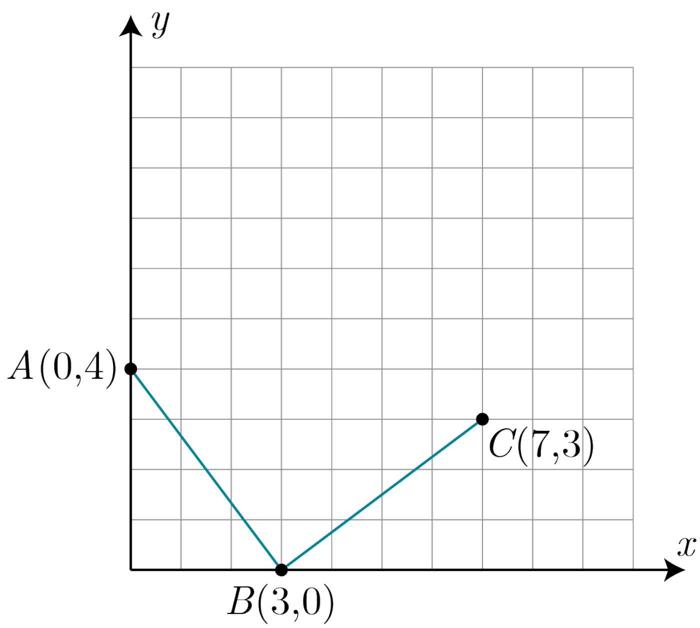
Suppose $\angle ADB = 30^\circ$ and $\angle CAD = x^\circ$. Show that $\angle ACB = 60^\circ$.

Problem of the Week

Problem C

Squared Up

Square $ABCD$ is drawn on graph paper with three of the vertices located at $A(0,4)$, $B(3,0)$, and $C(7,3)$, as shown.



Determine the area of square $ABCD$.

Theme: Geometry & Measurement

Problem of the Week
Problem C and Solution
Squared Up

Problem

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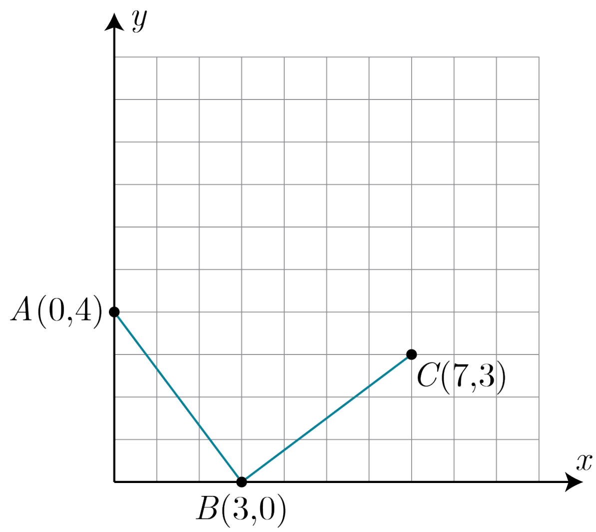
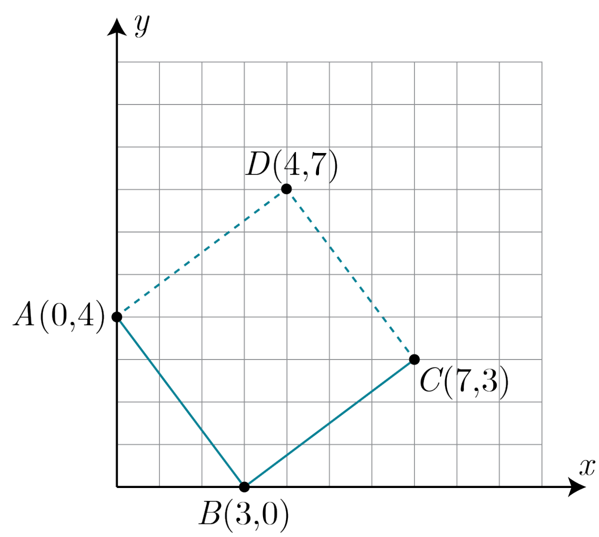


Determine the area of square $ABCD$.

Solution

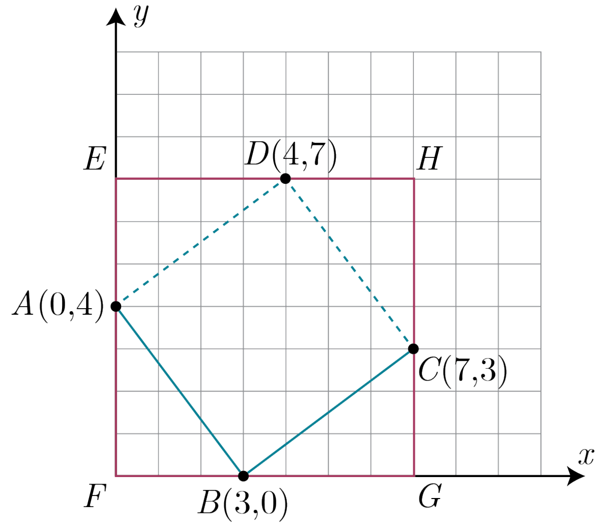
Solution 1

In this solution we start by determining the coordinates of the fourth vertex. To do this, we observe that to get from A to B we move 3 units right and 4 units down. Then, to get from B to C we move 4 units right and 3 units up. Continuing the pattern, to get from C to D we will move 3 units left and 4 units up to reach $D(4,7)$. As a check, to get from D to A we move 4 units left and 3 units down, as desired.



Alternatively you could use a protractor to draw sides AD and CD on grid paper, knowing that the corners of a square have right angles. This would also give you $D(4,7)$.

Next we draw rectangle $EFGH$ with horizontal and vertical sides around $ABCD$ so that each vertex of $ABCD$ lies on one of the sides of the rectangle. This rectangle ends up being a square with side length 7. Inside $EFGH$ is square $ABCD$ as well as four identical right-angled triangles, namely $\triangle AED$, $\triangle AFB$, $\triangle BGC$, and $\triangle CHD$. Each of these right-angled triangles has a base of 4 and a height of 3.



Finally, we calculate the area of $ABCD$. Since the triangles are all identical, their total area is equal to four times the area of any one of them.

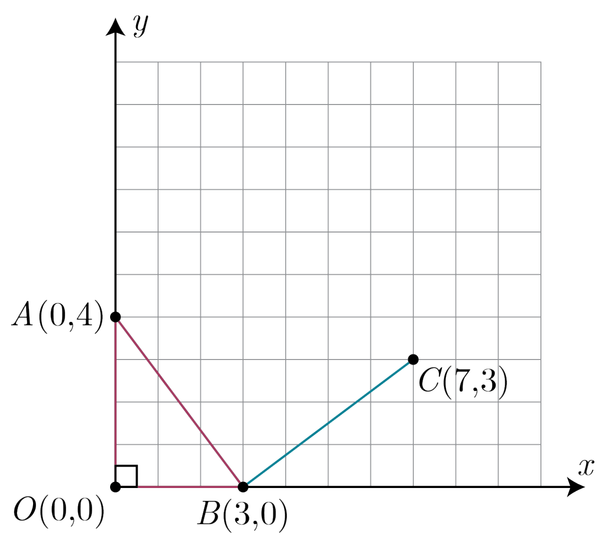
$$\begin{aligned}\text{Area } ABCD &= \text{Area } EFGH - 4 \times \text{Area } \triangle AED \\ &= 7 \times 7 - 4 \times (4 \times 3 \div 2) \\ &= 49 - 4 \times 6 \\ &= 49 - 24 = 25\end{aligned}$$

Therefore, the area of $ABCD$ is 25 units^2.

Solution 2

In this solution we will calculate the area of $ABCD$ without determining the coordinates of D. Instead we will use the Pythagorean Theorem.

Since $ABCD$ is a square, in order to calculate its area we need to find the length of only one of its sides. Let $O(0,0)$ be the point at the origin. Then $\triangle AOB$ is a right-angled triangle. The length of side OA is 4 and the length of side OB is 3. We can use the Pythagorean Theorem to calculate AB^2, which is the area of square $ABCD$.

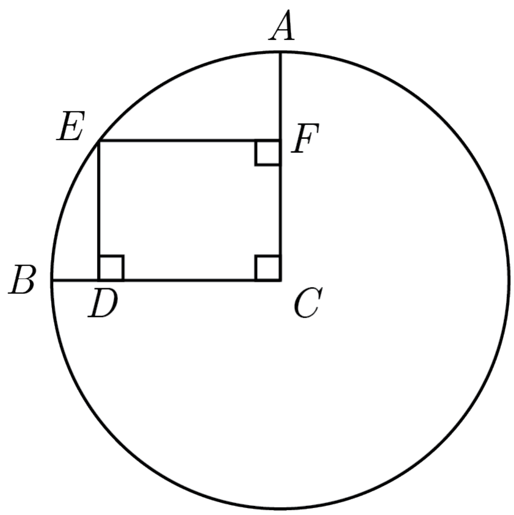


$$\begin{aligned}AB^2 &= OA^2 + OB^2 \\ &= 4^2 + 3^2 \\ &= 16 + 9 = 25\end{aligned}$$

Therefore, the area of $ABCD$ is 25 units^2.

Problem of the Week
Problem C
A Rectangle in a Circle

Points A and B are on the circumference of a circle with radius 10 cm and centre C such that $AC \perp BC$. Point D lies inside the circle on BC such that $BD = 2$ cm. Point E lies on the circumference of the circle, on the minor arc AB, such that $DE \perp BC$. Point F lies inside the circle on AC such that $CDEF$ forms a rectangle.

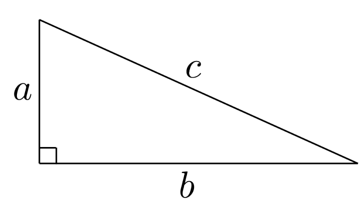


Determine the distance from A to F.

NOTE: You may find the following useful:

The *Pythagorean Theorem* states, "In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides."

In the right-angled triangle shown, c is the hypotenuse, a and b are the lengths of the other two sides, and $c^2 = a^2 + b^2$.



THEME: GEOMETRY & MEASUREMENT

Problem of the Week
Problem C and Solution
A Rectangle in a Circle

Problem

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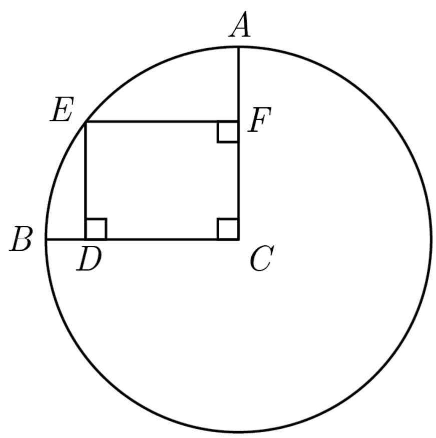

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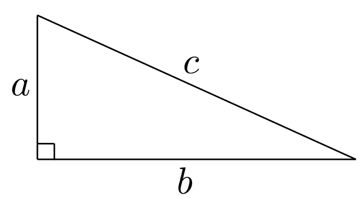
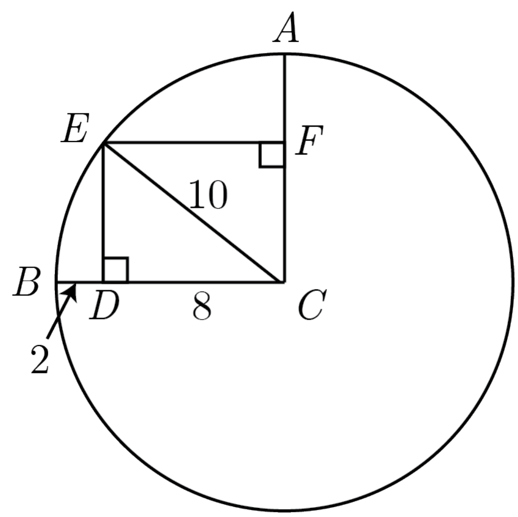


Solution

First we draw the radius EC. Since the radius of the circle is 10 cm, it follows that $AC = BC = EC = 10$. Then $DC = BC - BD = 10 - 2 = 8$.



Since $\angle CDE = 90^\circ$, it follows that $\triangle CDE$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle CDE$,

$$\begin{aligned} DE^2 + CD^2 &= EC^2 \\ DE^2 &= EC^2 - CD^2 \\ &= 10^2 - 8^2 \\ &= 100 - 64 \\ &= 36 \end{aligned}$$

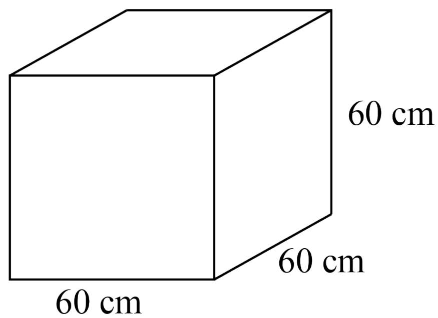
Then $DE = \sqrt{36} = 6$, since $DE > 0$.

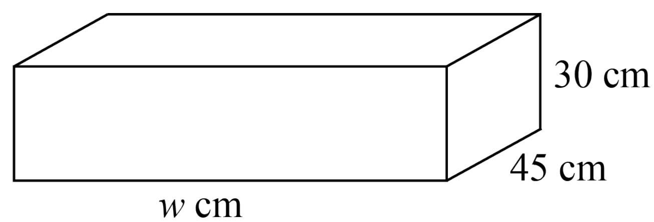
Since $CDEF$ is a rectangle, $CF = DE = 6$. Then
$AF = AC - CF = 10 - 6 = 4$. Thus, the distance from A to F is 4 cm.

Problem of the Week
Problem C
Hidden Space

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?





Themes: Algebra, Geometry & Measurement

Problem of the Week

Problem C and Solution

Hidden Space

Problem

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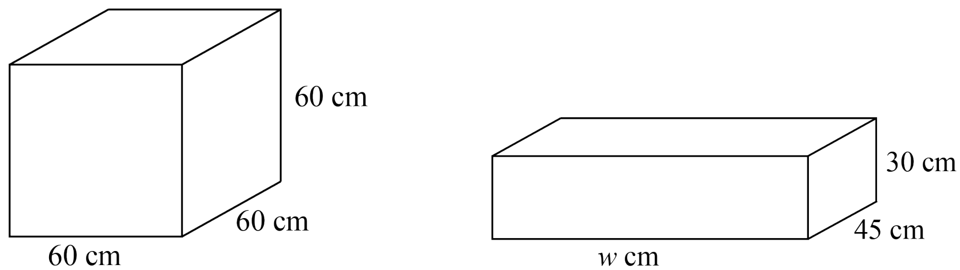


Solution

To determine the surface area of a cube, we determine the area of one side and multiply by 6. Therefore, the total surface area of the cube-shaped box is $6 \times 60 \times 60 = 21\,600$ cm^2.

To determine the total surface area of the rectangular prism, we determine the sum of the areas of the six sides. The front side and the back side have equal areas. The top and the bottom have equal areas. Each of the remaining two sides have equal area. Therefore, the total surface area of the rectangular prism box is

$$\begin{aligned} 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of side} &= 2 \times 30 \times w + 2 \times 45 \times w + 2 \times 30 \times 45 \\ &= 60w + 90w + 2700 \\ &= 150w + 2700 \end{aligned}$$

But the total surface area of the cube-shaped box is the same as the total surface area of the rectangular prism. Therefore,

$$\begin{aligned} 21\,600 &= 150w + 2700 \\ 21\,600 - 2700 &= 150w + 2700 - 2700 \\ 18\,900 &= 150w \\ \frac{18\,900}{150} &= \frac{150w}{150} \\ 126 &= w \end{aligned}$$

Therefore, the width of the rectangular prism is 126 cm.

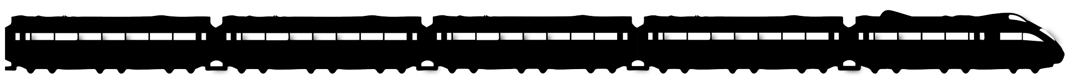
To determine the volume of a cube, we "cube" the edge length. So, the volume of the cube is $60 \times 60 \times 60 = 60^3 = 216\,000$ cm^3.

To determine the volume of the rectangular prism, we multiply the three different edge lengths. So, the volume of the rectangular prism is $126 \times 45 \times 30 = 170\,100$ cm^3.

Therefore, the student who is building the cube has the box with the greater volume. The volume of the cube is greater by $216\,000 - 170\,100 = 45\,900$ cm^3.

Problem of the Week
Problem C
In the Dark

A train that is 400 metres long approaches a tunnel that is 1200 metres long. The entire train is completely in the tunnel for 30 seconds. That is, from the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end of the tunnel, 30 seconds pass. If the train is travelling at a constant speed, determine this speed in kilometres per hour.



Themes: Geometry & Measurement, Number Sense

Problem of the Week

Problem C and Solution

In the Dark

Problem

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Solution

When the last car of the train has completely entered the tunnel, the front of the train is $1200 - 400 = 800$ metres from the other end of the tunnel. Thus, the train will travel 800 metres in 30 seconds. We can calculate the speed of the train by dividing the distance travelled by the time required to travel the distance.

The speed of the train is $800 \div 30 = \frac{80}{3}$ m/s.

Now we must convert from m/s to km/h. We will do this in two steps: first convert metres to kilometres, and then convert seconds to hours.

1. Metres to kilometres: $\dfrac{80 \text{ m}}{3 \text{ s}} = \dfrac{80 \text{ m}}{3 \text{ s}} \times \dfrac{1 \text{ km}}{1000 \text{ m}} = \dfrac{2 \text{ km}}{75 \text{ s}}$
2. Seconds to hours: $\dfrac{2 \text{ km}}{75 \text{ s}} = \dfrac{2 \text{ km}}{75 \text{ s}} \times \dfrac{60 \text{ s}}{1 \text{ min}} \times \dfrac{60 \text{ min}}{1 \text{ hr}} = \dfrac{96 \text{ km}}{1 \text{ hr}}$

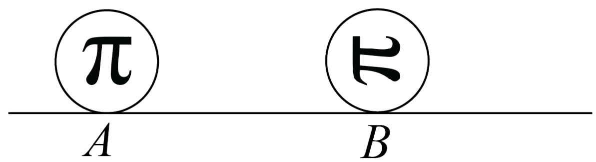
Therefore, the train is travelling at a speed of 96 km/hr.

Problem of the Week

Problem C

Pi Day is Rolling Along

A coin has the image of π on one side and is being rolled along a horizontal surface. It starts at position A, with the π symbol oriented correctly. The coin then rolls until it reaches position B, which is 9 cm to the right of A. If the image of π is now rotated 90° counterclockwise for the first time, then what is the radius of the coin? Round your answer to 1 decimal. What orientation will the image of π be in after it has rolled a total of 174 cm?



Themes: Algebra, Geometry & Measurement

Problem of the Week
Problem C and Solution
Pi Day is Rolling Along

Problem

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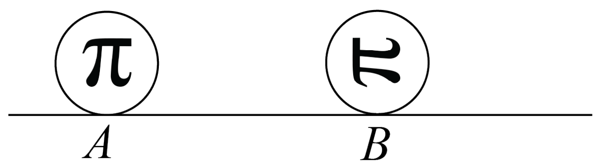


Solution

At position B, the coin has rotated 90° counterclockwise for the first time. Thus, it must have gone through $\frac{3}{4}$ of a full rotation. Since the distance from A to B is 9 cm and the coin has gone through $\frac{3}{4}$ of a full rotation, a full rotation would move the coin 12 cm along the line from position A.

A full rotation is also completed by travelling one measure of the circumference of the coin.

Therefore, the circumference of the coin is equal to 12 cm.

We also know that $C = 2 \times \pi \times r$, where C is the circumference of a circle and r is its radius.

Thus, $12 = 2 \times \pi \times r$. Solving, we have $r = \dfrac{12}{2\pi} = \dfrac{6}{\pi} \approx 1.9$ cm.

Since the coin completes a full rotation every 12 cm, after it coin has rolled 174 cm along the line, it would have completed $\frac{174}{12} = 14.5$ revolutions.

This means the coin has completed 14 full revolutions and one-half of a revolution. Thus, the image of π will be upside down.

Problem of the Week

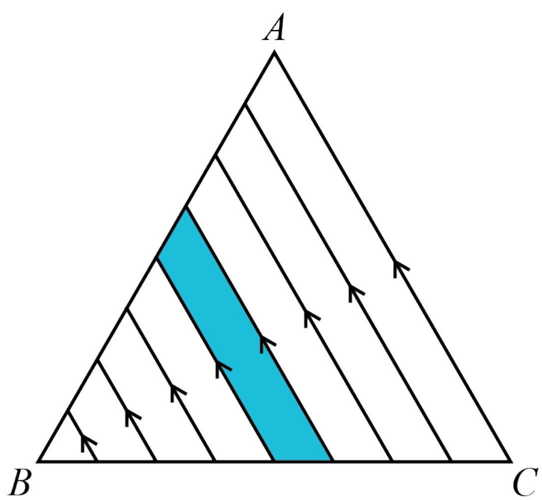
Problem C

Suncatcher

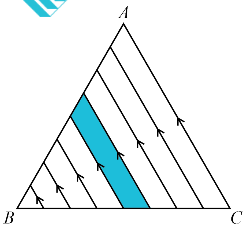
A glass suncatcher is in the shape of an equilateral triangle with sides of length 144 mm. The triangle is labeled ABC and divided into 8 smaller sections as follows.

- Sides AB and BC are each divided into 8 segments of equal length.
- Each point of division on AB is connected to its corresponding point of division on BC, creating 7 line segments.
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

One of the sections is coloured blue, as shown. Determine the perimeter of this section.



Theme: Geometry & Measurement



Problem of the Week
Problem C and Solution
Suncatcher

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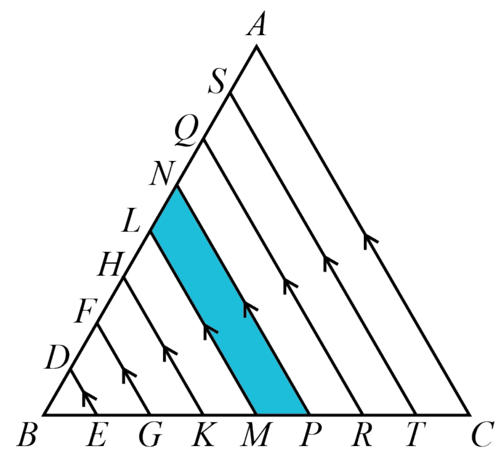
- Each of the 7 line segments is parallel to the third side of the triangle, AC.

One of the sections is coloured blue, as shown. Determine the perimeter of this section.

Solution

Since AB and BC have each been divided into 8 equal segments, we will label the points of division as shown. Then, since $AB = BC = 144$ mm, and $144 \div 8 = 18$, it follows that $BD = DF = FH = HL = LN = NQ = QS = SA = 18$ mm, and $BE = EG = GK = KM = MP = PR = RT = TC = 18$ mm.



We now proceed with two different solutions.

Solution 1

The angles in an equilateral triangle are each 60°. Therefore,
$\angle BAC = \angle BCA = \angle ABC = 60°$.

Since $LM \parallel NP \parallel AC$, $\angle BLM = \angle BNP = \angle BAC = 60°$ and
$\angle BML = \angle BPN = \angle BCA = 60°$.

In $\triangle BLM$, $\angle BLM = \angle BML = \angle LBM = 60°$ and it follows that $\triangle BLM$ is equilateral. Since $BL = BD + DF + FH + HL = 18 + 18 + 18 + 18 = 72$, it follows that $LM = 72$ mm.

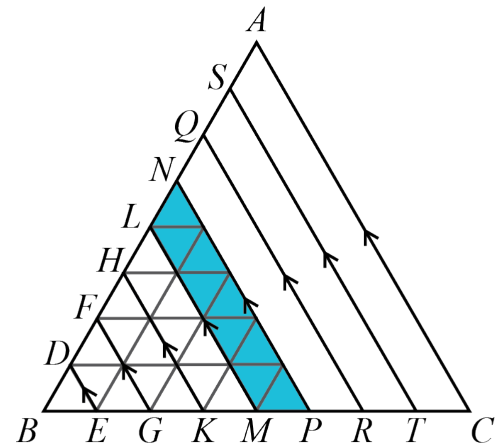
Similarly, in $\triangle BNP$, $\angle BNP = \angle BPN = \angle NBP = 60°$ and it follows that $\triangle BNP$ is equilateral. Since $BN = BD + DF + FH + HL + LN = 5 \times 18 = 90$, it follows that $NP = 90$ mm.

Therefore the perimeter of the shaded region is
$LM + MP + NP + LN = 72 + 18 + 90 + 18 = 198$ mm.

Solution 2

Observe that $\triangle ABC$ can be tiled with small equilateral triangles congruent to $\triangle BDE$. That is, equilateral triangles with side length 18 mm. A complete justification of this is not provided here but you may wish to verify this for yourself.

Three of the small equilateral triangles cover the entire area occupied by quadrilateral $DEGF$. Five of the small equilateral triangles cover the entire area occupied by quadrilateral $FGKH$. Seven of the small equilateral triangles cover the entire area occupied by quadrilateral $HKML$. Nine of the small equilateral triangles cover the entire area occupied by quadrilateral $LMPN$.



If we were to continue, $\triangle ABC$ would contain $1+3+5+7+9+11+13+15=64$ of the small equilateral triangles.

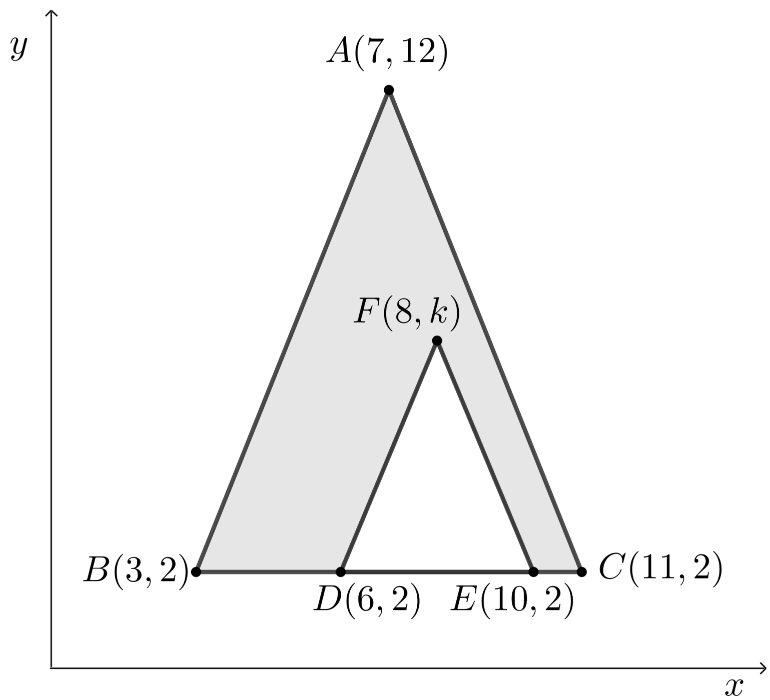
In quadrilateral $LMPN$, the smaller side, LM, contains the bases of 4 of the small equilateral triangles and therefore is $4 \times 18 = 72$ mm long. The larger side, NP, contains the bases of 5 of the small equilateral triangles and therefore is $5 \times 18 = 90$ mm long.

The perimeter is the sum of the lengths of the four sides of quadrilateral $LMPN$. Therefore, the perimeter of the shaded region is
$LM + MP + NP + LN = 72 + 18 + 90 + 18 = 198$ mm.

Problem of the Week
Problem C
Tangled Triangles

Points $A(7,12)$, $B(3,2)$, $C(11,2)$, $D(6,2)$ and $E(10,2)$ are plotted on the Cartesian plane. The point $F(8,k)$ lies inside $\triangle ABC$ so that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2.

Determine the value of k, the y-coordinate of F.



Themes: Algebra, Geometry & Measurement

Problem of the Week
Problem C and Solution
Tangled Triangles

Problem

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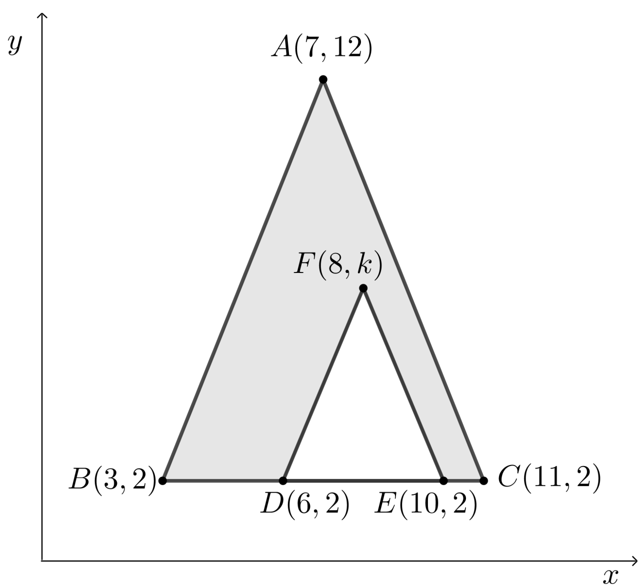


Solution

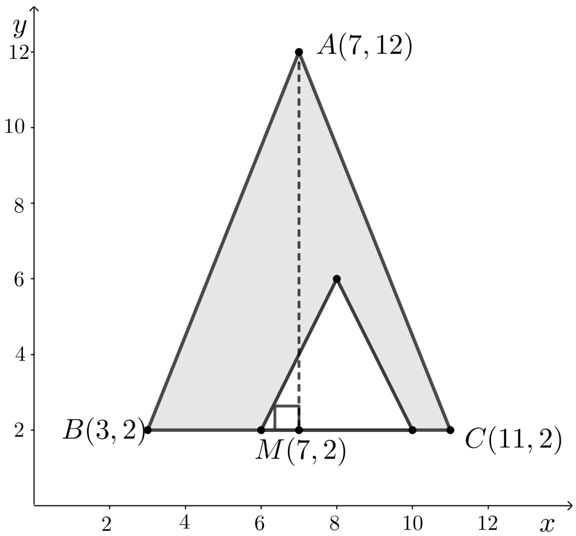
We will calculate the area of $\triangle ABC$, calculate the area of $\triangle DEF$, and then use the given information that the difference between these areas is equal to 32 units^2.

We will use the fact that the distance between two points that have the same x-coordinate is the positive difference between their y-coordinates. We will also use the fact that the distance between two points that have the same y-coordinate is the positive difference between their x-coordinates.

Since B and C both have y-coordinate 2, BC is a horizontal line. Thus, $BC = 11 - 3 = 8$.

In $\triangle ABC$, drop a perpendicular from vertex A to M on BC. Since BC is horizontal, then AM is vertical. Since every point on a vertical line has the same x-coordinate, M has x-coordinate 7. Similarly, since M is on the horizontal line through $B(3, 2)$ and $C(11, 2)$, M has y-coordinate 2. Therefore, $AM = 12 - 2 = 10$.

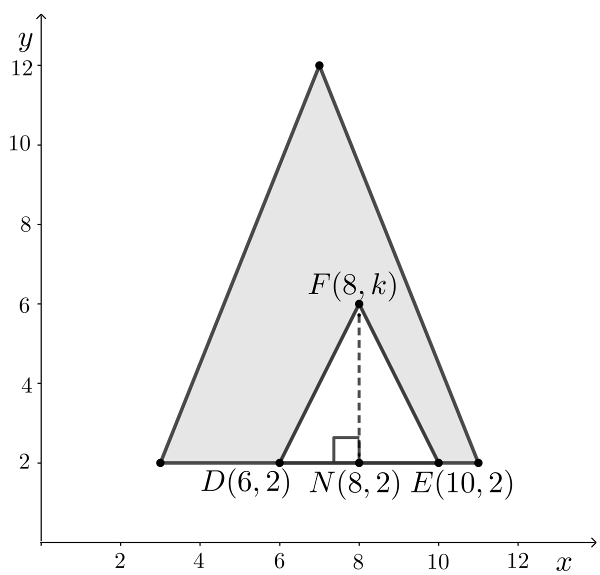
Thus, the area of $\triangle ABC$ is equal to $\dfrac{BC \times AM}{2} = \dfrac{8 \times 10}{2} = 40$.



Since D and E both have y-coordinate 2, DE is a horizontal line. Thus, $DE = 10 - 6 = 4$.

In $\triangle DEF$, drop a perpendicular from vertex F to N on DE. Since DE is horizontal, then FN is vertical. Since every point on a vertical line has the same x-coordinate, N has x-coordinate 8. Similarly, since N is on the horizontal line through $D(6, 2)$ and $E(10, 2)$, N has y-coordinate 2. Therefore, $FN = k - 2$.

Thus, the area of $\triangle DEF$ is equal to $\dfrac{DE \times FN}{2} = \dfrac{4 \times (k-2)}{2} = 2(k-2)$.



We now use the given information that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2 to obtain the equation $40 - 2(k-2) = 32$.

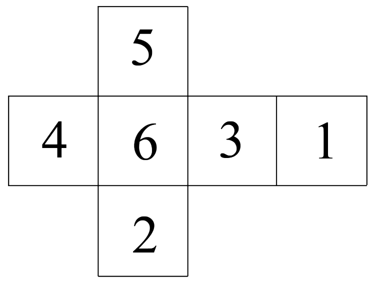
Subtracting 32 from each side, we have $8 - 2(k-2) = 0$.

Adding $2(k-2)$ to each side, we have $8 = 2(k-2)$, which simplifies to $4 = k - 2$, or $k = 6$.

Therefore, the value of k, the y-coordinate of point F, is 6.

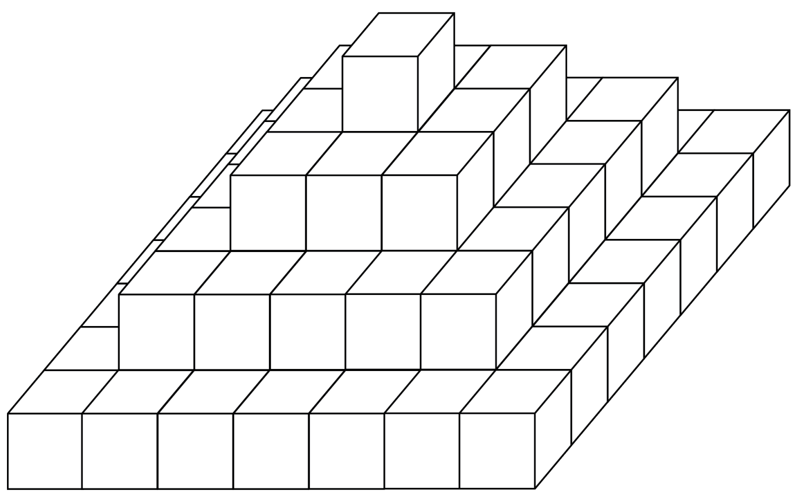
Problem of the Week
Problem C
Counting Numbered Cubes

Dmitri has a collection of identical cubes. Each cube is labelled with the integers 1 to 6, as shown in the following net:



(This net can be folded to make a cube.)

He forms a pyramid by stacking layers of the cubes on a table, as shown, with the bottom layer being a 7 by 7 square of cubes.



(a) Determine the total number of cubes used to build the pyramid.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) He wants to position the cubes so that when all of the visible numbers are added up, the total is as large as possible. What is this total?

Themes: Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Counting Numbered Cubes

Problem

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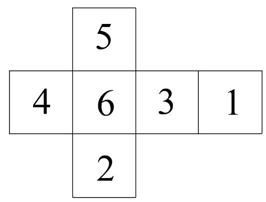

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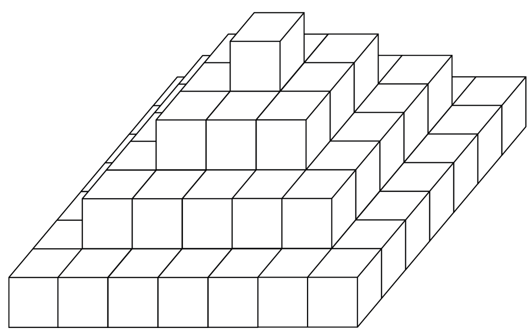


(a) Determine the total number of cubes used to build the pyramid.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) He wants to position the cubes so that when all of the visible numbers are added up, the total is as large as possible. What is this total?

Solution

(a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses $7 \times 7 = 49$ cubes.
The next layer of the cube is a 5 by 5 square of cubes, so uses $5 \times 5 = 25$ cubes.
The next layer of the cube is a 3 by 3 square of cubes, so uses $3 \times 3 = 9$ cubes.
The top layer consists of a single cube.
Therefore, the total number of cubes used is $49 + 25 + 9 + 1 = 84$.

(b) **Solution 1**

The cube in the top layer has 5 visible faces (only the bottom face is hidden).

In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).

In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 3 \times 2 = 24$ visible faces).

In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 5 \times 2 = 40$ visible faces).

Therefore, the total number of visible faces is
$5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$.

Solution 2

When we look at the pyramid from the top, we see a 7 by 7 square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)

When we look at the pyramid from each of the four sides, we see
$1 + 3 + 5 + 7 = 16$ visible faces, so there are $4(16) = 64$ visible faces on the sides.

Therefore, there are $49 + 64 = 113$ visible faces in total.

(c) **Solution 1**

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the "1" on the bottom face (and so is hidden). The total of the numbers visible on this cube is $2 + 3 + 4 + 5 + 6 = 20$.

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5, and 6 all visible (this is possible since the faces with the 4, 5, and 6 share a vertex) and the 1, 2, and 3 hidden.

There are 12 of these cubes, so the total of the numbers visible on these cubes is $12 \times (4 + 5 + 6) = 180$.

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.

There are $4 + 12 + 20 = 36$ of these cubes, so the total of the numbers visible on these cubes is $36 \times (5 + 6) = 396$.

Therefore, the overall largest possible total is $20 + 180 + 396 = 596$.

Solution 2

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top face which is visible is 6 (for a total of $49 \times 6 = 294$).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds $2 + 3 + 4 + 5 = 14$ to the total of the visible faces thus far.

Consider lastly the faces visible on the sides (a total of $4 \times (3 + 5 + 7) = 60$ faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s).

This would give a total of $5 \times 60 = 300$.

However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5, and 6 meet at a vertex on each cube.)

Therefore, the overall largest possible total is $294 + 14 + 300 - 12 = 596$.

Number Sense (N)

Take me to the cover

Problem of the Week

Problem C

Sweet Treats

Nadia can confidently make 32 different desserts. Sixteen of the desserts contain gluten and twelve contain dairy. Seven of the desserts contain both gluten and dairy. How many of the desserts are both gluten free and dairy free?

Themes: Data Management, Number Sense

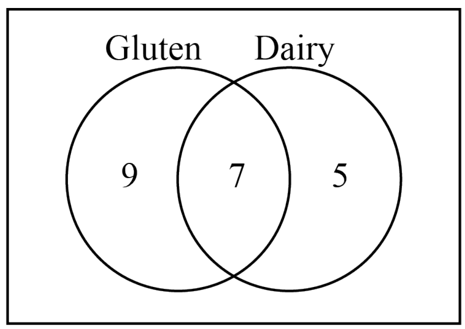
Problem of the Week
Problem C and Solution
Sweet Treats

Problem

Nadia can confidently make 32 different desserts. Sixteen of the desserts contain gluten and twelve contain dairy. Seven of the desserts contain both gluten and dairy. How many of the desserts are both gluten free and dairy free?

Solution

Since 7 desserts contain both gluten and dairy, and these are included in the 16 desserts that contain gluten, then $16 - 7 = 9$ desserts contain gluten but not dairy. Similarly, the 7 desserts that contain both gluten and dairy are included in the 12 desserts that contain dairy, so $12 - 7 = 5$ desserts contain dairy but not gluten. This is shown in the following Venn diagram.



The desserts can then be separated into four categories: desserts that contain gluten but not dairy, desserts that contain dairy but not gluten, desserts that contain both dairy and gluten, and desserts that are both gluten free and dairy free. If we add the number of desserts in each category, we will get the total number of desserts that Nadia can make, which is 32. Thus, the number of desserts that are both gluten free and dairy free is equal to the total number of desserts minus the number of desserts that contain gluten but not dairy minus the number of desserts that contain dairy but not gluten minus the number of desserts that contain both gluten and dairy.

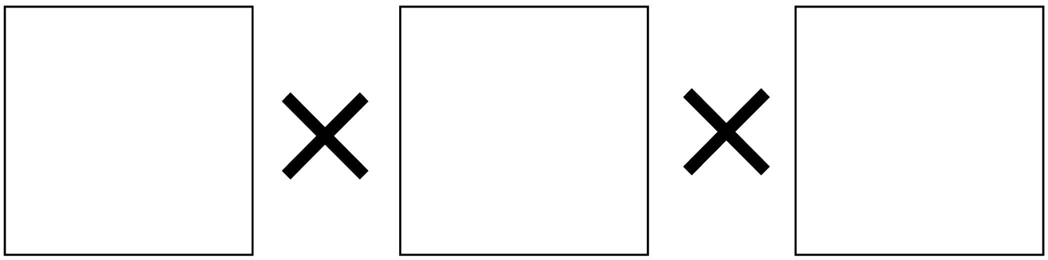
Since $32 - 9 - 5 - 7 = 11$, it follows that the number of desserts that are both gluten free and dairy free is 11.

Problem of the Week

Problem C

Multiply to the Max

Sophia selects three different numbers from the set $\{-7, -5, -3, -1, 0, 2, 4, 6, 8\}$. She then determines the product of the three numbers. What is the largest product that Sophia can make?



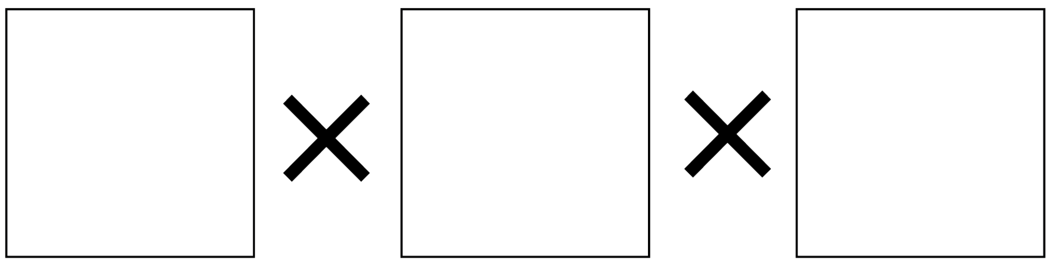
Problem of the Week

Problem C and Solution

Multiply to the Max

Problem

Sophia selects three different numbers from the set $\{-7, -5, -3, -1, 0, 2, 4, 6, 8\}$. She then determines the product of the three numbers. What is the largest product that Sophia can make?



Solution

Since $4 \times 6 \times 8 = 192$, then the largest possible product is *at least* 192. In particular, the largest possible product is positive.

For the product of three numbers to be positive, either all three numbers are positive or one number is positive and two numbers are negative. (If there were an odd number of negative factors, the product would be negative.)

If all three numbers are positive, the product is as large as possible when the three numbers are each as large as possible. In this case, the largest possible product is $4 \times 6 \times 8 = 192$.

If one number is positive and two numbers are negative, their product is as large as possible if the positive number is as large as possible, so is 8, and the product of the two negative numbers is as large as possible.

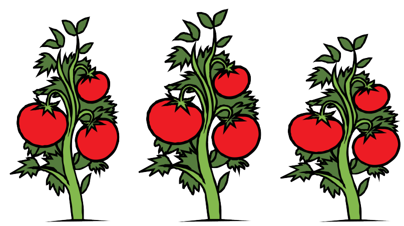
The product of the two negative numbers will be as large as possible when the negative numbers are each as far from 0 as possible. These numbers are thus -5 and -7, with product $(-5) \times (-7) = 35$. (We can check the other possible products of two negative numbers and see that none is as large.)

So the largest possible product in this case is $8 \times (-5) \times (-7) = 8 \times 35 = 280$.

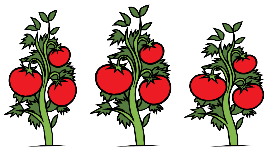
Combining the two cases, we find that the largest possible product is 280.

Problem of the Week
Problem C
Prime Tomatoes

Antoine has three tomato plants. The total number of tomatoes on all his plants is 41, and one of his plants has exactly 4 more tomatoes than another plant. If the number of tomatoes on each plant is a prime number, determine all possibilities for the number of tomatoes on each plant.



Note: A *prime number* is an integer greater than 1 that has only two positive factors: 1 and itself. The number 17 is prime because its only positive factors are 1 and 17.



Problem of the Week
Problem C and Solution
Prime Tomatoes

Problem

Antoine has three tomato plants. The total number of tomatoes on all his plants is 41, and one of his plants has exactly 4 more tomatoes than another plant. If the number of tomatoes on each plant is a prime number, determine all possibilities for the number of tomatoes on each plant.

Note: A *prime number* is an integer greater than 1 that has only two positive factors: 1 and itself. The number 17 is prime because its only positive factors are 1 and 17.

Solution

This problem is asking us to find three prime numbers with a sum of 41, where two of the prime numbers differ by exactly 4. We can start by listing all the prime numbers less than 41. They are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37.

Next we can look for all pairs of numbers from this list that differ by 4. These pairs are as follows: 3 and 7, 7 and 11, 13 and 17, and 19 and 23.

Since we want three prime numbers that add to 41, we can look at each of these pairs of numbers to determine all possible solutions.

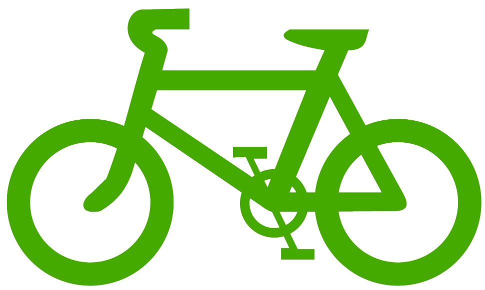
- 3 and 7: The third number would be $41 - 3 - 7 = 31$. Since this is a prime number, this is a possible solution.
- 7 and 11: The third number would be $41 - 7 - 11 = 23$. Since this is a prime number, this is a possible solution.
- 13 and 17: The third number would be $41 - 13 - 17 = 11$. Since this is a prime number, this is a possible solution.
- 19 and 23: Since $19 + 23 = 42$, this sum is already over 41, so this is not a possible solution.

Therefore, there are three solutions. The tomato plants could have 3, 7, and 31 tomatoes each, 7, 11, and 23 tomatoes each, or 13, 17, and 11 tomatoes each.

Problem of the Week
Problem C
Bicycle Trip

Cy Kler has mapped out a 560 km bike route that he wants to complete in seven days. Each day he wants to ride 15 km more than the day before.

If Cy is able to follow his plan, then how many kilometres will he have to ride on the seventh day of the trip?



Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Bicycle Trip

Problem

Cy Kler has mapped out a 560 km bike route that he wants to complete in seven days. Each day he wants to ride 15 km more than the day before.

If Cy is able to follow his plan, then how many kilometres will he have to ride on the seventh day of the trip?

Solution

Solution 1

We consider how much extra Cy Kler rides on each day after the first.

On Day 2, he rides 15 km extra.
On Day 3, he rides $15 + 15 = 30$ km extra.
On Day 4, he rides $15 + 15 + 15 = 45$ km extra.
On Day 5, he rides $15 + 15 + 15 + 15 = 60$ km extra.
On Day 6, he rides $15 + 15 + 15 + 15 + 15 = 75$ km extra.
On Day 7, he rides $15 + 15 + 15 + 15 + 15 + 15 = 90$ km extra.

Thus, the total distance is made up of seven day trips, each day beginning with the same distance as Day 1 plus $15 + 30 + 45 + 60 + 75 + 90 = 315$ km extra.

So, seven days of riding the same distance as Day 1 would total $560 - 315 = 245$ km. Thus, on Day 1, Cy Kler will ride $245 \div 7 = 35$ km.

On Day 7, Cy Kler will ride the Day 1 distance plus 90 km. Thus, the distance that he will ride on the seventh day is $35 + 90 = 125$ km.

Solutions 2 and 3 present more algebraic approaches to this problem.

Solution 2

Let x be the distance Cy Kler will ride on Day 1. Then he will ride $x+15$, $x+30$, $x+45$, $x+60$, $x+75$, and $x+90$ km on Day 2 through Day 7, respectively. Then,

$$\begin{aligned}
x+(x+15)+(x+30)+(x+45)+(x+60)+(x+75)+(x+90) &= 560\\
7x+15+30+45+60+75+90 &= 560\\
7x+315 &= 560\\
7x+315-315 &= 560-315\\
7x &= 245\\
\frac{7x}{7} &= \frac{245}{7}\\
x &= 35
\end{aligned}$$

Thus, Cy Kler will ride 35 km on Day 1. Thus, the distance that he will ride on the seventh day is $x+90=35+90=125$ km.

Solution 3

Let m be the distance Cy will ride on Day 4, the middle day. On Day 5 he would ride $(m+15)$ km, on Day 6 he would ride $m+15+15=(m+30)$ km, and on Day 7 he would ride $m+30+15=(m+45)$ km. Working backwards from Day 4, we reduce the distance he rides by 15 km. On Day 3 he would ride $(m-15)$ km, on Day 2 he would ride $m-15-15=(m-30)$ km, and on Day 1 he would ride $m-30-15=(m-45)$ km. Then,

$$\begin{aligned}
m+(m+15)+(m+30)+(m+45)+(m-15)+(m-30)+(m-45) &= 560\\
7m+15+30+45-15-30-45 &= 560\\
7m+15-15+30-30+45-45 &= 560\\
7m &= 560\\
\frac{7m}{7} &= \frac{560}{7}\\
m &= 80
\end{aligned}$$

Thus, Cy Kler will ride 80 km on Day 4. Thus, the distance that he will ride on the seventh day is $m+45=80+45=125$ km.

A solution like Solution 3 works well when there are an odd number of terms in a sequence that increases (or decreases) by a constant amount.

Problem of the Week
Problem C
Math Test Math 1

Dilip has had six tests in his math class so far. He knows the average (mean) of some of his marks.

- The average of his first and second test marks is 77%.
- The average of his third and fourth test marks is 75%.
- The average of his fifth and sixth test marks is 82%.

Determine the average of all six test marks for Dilip.

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Math Test Math 1

Problem

Dilip has had six tests in his math class so far. He knows the average (mean) of some of his marks.

- The average of his first and second test marks is 77%.
- The average of his third and fourth test marks is 75%.
- The average of his fifth and sixth test marks is 82%.

Determine the average of all six test marks for Dilip.

Solution

The average (mean) of all six test marks is equal to the sum of all six test marks divided by 6.

Since the average of Dilip's first and second test marks is 77, it follows that the sum of the first and second test marks is $2 \times 77 = 154$. Similarly, the sum of his third and fourth test marks is $2 \times 75 = 150$, and the sum of his fifth and sixth test marks is $2 \times 82 = 164$. It follows that the sum of all six test marks is $154 + 150 + 164 = 468$. Since $468 \div 6 = 78$, it follows that the average of all six test marks is 78%.

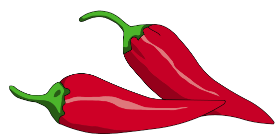
Problem of the Week

Problem C

Needs More Heat

Samu bought 20 g of curry powder in a container. He used 2 g of the curry powder in a recipe and thought it wasn't spicy enough, so he added 2 g of hot chili powder to his curry powder container and mixed it thoroughly. The next day he used 2 g of his newly mixed curry powder in a recipe and thought it still wasn't spicy enough, so he added another 2 g of hot chili powder to his curry powder container and mixed it thoroughly.

What is the ratio of the mass of the original curry powder to the mass of hot chili powder in the container now?



Theme: Number Sense

Problem of the Week

Problem C and Solution

Needs More Heat

Problem

Samu bought 20 g of curry powder in a container. He used 2 g of the curry powder in a recipe and thought it wasn't spicy enough, so he added 2 g of hot chili powder to his curry powder container and mixed it thoroughly. The next day he used 2 g of his newly mixed curry powder in a recipe and thought it still wasn't spicy enough, so he added another 2 g of hot chili powder to his curry powder container and mixed it thoroughly.

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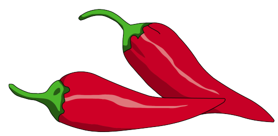


Solution

After Samu used the curry powder the first time, there was $20 - 2 = 18$ g of the curry powder left. Samu then added 2 g of hot chili powder, so the container had 18 g of the original curry powder and 2 g of hot chili powder.

After Samu used his newly mixed curry powder, there was $20 - 2 = 18$ g of the mixture left. We need to determine how much of this is the original curry powder and how much is hot chili powder. Of the mixture, $\frac{18}{20}$ or $\frac{9}{10}$ is the original curry powder and the remainder is hot chili powder. So if there is 18 g of this mixture, then $\frac{9}{10} \times 18 = 16.2$ g is the original curry powder and the remaining $18 - 16.2 = 1.8$ g is hot chili powder.

After Samu added another 2 g of hot chili powder, the container had 16.2 g of the original curry powder and $1.8 + 2 = 3.8$ g of hot chili powder. Thus, the ratio of the mass of the original curry powder to the mass of hot chili powder is

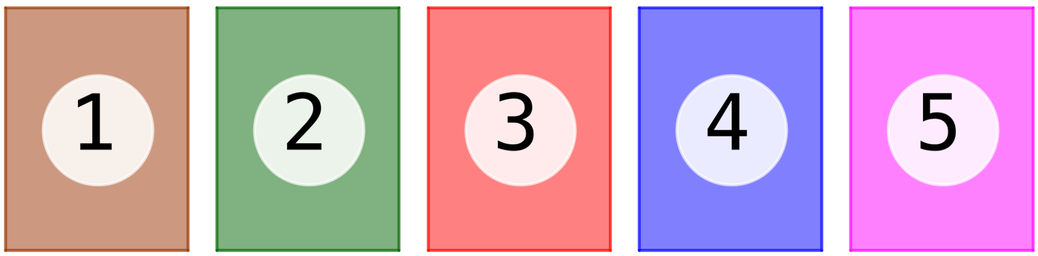
$$16.2 : 3.8 = 162 : 38 = 81 : 19$$

Problem of the Week

Problem C

Three Card Sum

Five cards are numbered with the integers 1 through 5. The cards are mixed up and placed face down on a table. A person randomly flips over three cards. What is the probability that the sum of the numbers on those three cards is odd?



Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Three Card Sum

Problem

Five cards are numbered with the integers 1 through 5. The cards are mixed up and placed face down on a table. A person randomly flips over three cards. What is the probability that the sum of the numbers on those three cards is odd?

Solution

In order to determine the probability, we must determine the number of ways to obtain a sum that is odd and divide it by the total number of possibilities for the three flipped cards. We will count the possibilities for the three flipped cards by systematically working through cases.

- Case 1: They flip 1 and 2 and one higher number. Then there are 3 possibilities for the higher number: 3, 4, and 5.
- Case 2: They flip 1 and 3 and one higher number. Then there are 2 possibilities for the higher number: 4 and 5.
- Case 3: They flip 1 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.
- Case 4: They flip 2 and 3 and one higher number. Then there are 2 possibilities for the higher number: 4 and 5.
- Case 5: They flip 2 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.
- Case 6: They flip 3 and 4 and one higher number. Then there is 1 possibility for the higher number: 5.

Therefore, there are $3 + 2 + 1 + 2 + 1 + 1 = 10$ possible flips of three cards. How many of these have an odd sum? We could take each of the possibilities, determine the sum, and then count the number which produce an odd sum. We will present a different method.

The sum of three integers is odd in two instances: all three integers are odd or one integer is odd and the other two are even.

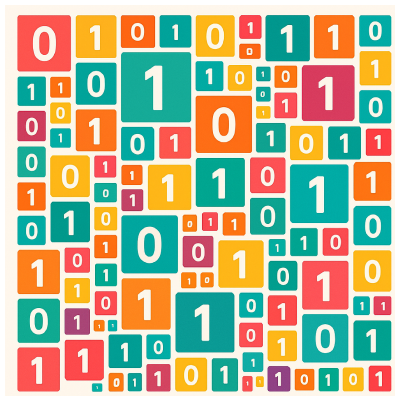
There is only one possibility with three odd integers. This is when 1, 3, and 5 are flipped. There are three possibilities with one odd integer and two even integers. They are when 1, 2, and 4 are flipped, or when 2, 3, and 4 are flipped, or when 2, 4, and 5 are flipped. Therefore, there are $1 + 3 = 4$ flips where the sum is odd.

Therefore, the probability of flipping three cards with an odd sum is $\frac{4}{10} = \frac{2}{5}$.

Extension: If nine cards numbered with the integers 1 through 9 are mixed up and placed face down on a table, then the probability of flipping over three cards that have an odd sum is $\frac{10}{21}$. Can you verify this?

Problem of the Week
Problem C
Ones and Zeros

Determine the three smallest positive integers that contain only 0s and 1s as digits and are divisible by 6.

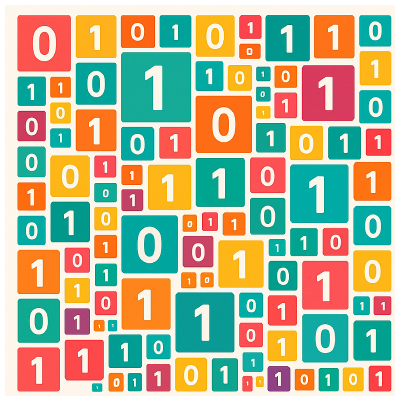


Theme: Number Sense

Problem of the Week
Problem C and Solution
Ones and Zeros

Problem

Determine the three smallest positive integers that contain only 0s and 1s as digits and are divisible by 6.



Solution

A number divisible by 6 must be divisible by both 2 and 3.

Since the number is divisible by 2, it must be even. Since the only digits in our number are 1s and 0s, the units digit must be 0.

If a number is divisible by 3, the sum of the digits must also be divisible by 3. Since the only digits in our number are 1s and 0s, the smallest positive integer divisible by 3 must contain three 1s.

To be as small as possible, the number must contain as few digits as possible. Therefore, the smallest such number contains three 1s and ends in a 0. This number is 1110.

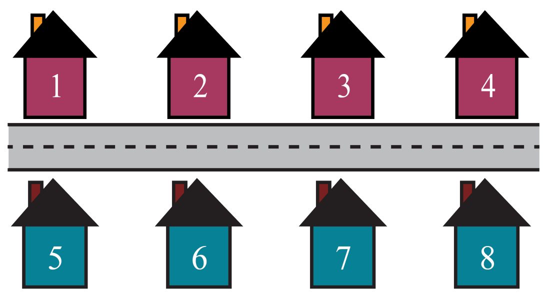
There are no more four-digit positive integers that end in a 0 and contain three 1s. Therefore, the next smallest number will have five digits. To be divisible by 3 and contain five digits, which are only 1s and 0s, the number must contain three 1s and two 0s. As stated above, one 0 must be the units digit. To be as small as possible, the second 0 must be in the highest place value possible. Therefore, the next two smallest positive integers are 10110, and 11010.

Thus, the three smallest positive integers to meet the requirements are 1110, 10110, and 11010.

Problem of the Week
Problem C
Across the Lane

Lookover Lane divides two rows of houses. Each house on one side of the lane is directly opposite a house on the other side of the lane. The houses are numbered consecutively 1, 2, 3, and so on, along one side. Once the end of that side of the lane is reached, the consecutive numbering continues at the house on the other side of the street opposite the house numbered 1. The consecutive numbering then continues along this second side until the last house is numbered.

For example, if there were eight houses on the lane, then House 1 would be opposite House 5, House 2 would be opposite House 6, House 3 would be opposite House 7, and House 4 would be opposite the House 8.



However, on the actual lane, House 37 is opposite House 2026. How many houses are on Lookover Lane?

Theme: Number Sense

Problem of the Week

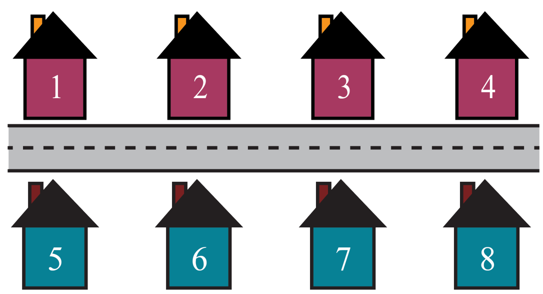
Problem C and Solution

Across the Lane

Problem

Lookover Lane divides two rows of houses. Each house on one side of the lane is directly opposite a house on the other side of the lane. The houses are numbered consecutively 1, 2, 3, and so on, along one side. Once the end of that side of the lane is reached, the consecutive numbering continues at the house on the other side of the street opposite the house numbered 1. The consecutive numbering then continues along this second side until the last house is numbered.

For example, if there were eight houses on the lane, then House 1 would be opposite House 5, House 2 would be opposite House 6, House 3 would be opposite House 7, and House 4 would be opposite the House 8.



However, on the actual lane, House 37 is opposite House 2026. How many houses are on Lookover Lane?

Solution

There are 36 houses before House 37 on the one side of the lane. Therefore, there must be 36 houses on the other side of the lane before House 2026.

So, the first house on the other side of the lane is House $2026 - 36 = 1990$. Therefore, the last house on the first side of the lane is House 1989. Each house on one side has a house directly across from it on the other side. Since there are 1989 houses on one side, there are 1989 houses on the other side. Thus, there are a total of $1989 \times 2 = 3978$ houses on Lookover Lane.

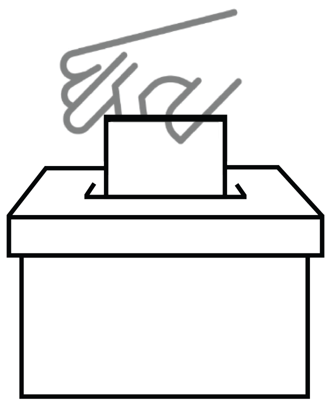
Problem of the Week

Problem C

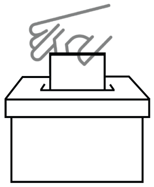
Your Vote Counts

In an election for a representative on school council, there were only two candidates.

Sophia received 60% of the total votes and Mireille received all the rest. If Sophia won by 28 votes, then how many people voted?



Themes: Algebra, Number Sense



Problem of the Week

Problem C and Solution

Your Vote Counts

Problem

In an election for a representative on school council, there were only two candidates.

Sophia received 60% of the total votes and Mireille received all the rest. If Sophia won by 28 votes, then how many people voted?

Solution

Solution 1

If Sophia received 60% of the votes, then Mireille received 40% of the total number of votes. The difference between them, 20%, represents 28 votes.

We are interested in determining 100% of the votes, that is, the total number of votes cast. Since we know that 20% of the total votes cast was 28 votes and $5 \times 20 = 100$, then a total of 5×28 or 140 people voted.

Solution 2

The second solution uses an algebraic approach.

Let n represent the total number of votes cast.

Since Sophia received 60% of the total votes, she received $0.6 \times n$ or $0.6n$ votes.

Since Mireille received all of the remaining votes, she received $0.4 \times n$ or $0.4n$ votes.

The difference between the number of votes received by Sophia and the number of votes received by Mireille was 28. So,

$$\begin{aligned} 0.6n - 0.4n &= 28 \\ 0.2n &= 28 \\ \frac{0.2n}{0.2} &= \frac{28}{0.2} \\ n &= 140 \end{aligned}$$

Therefore, 140 people voted.

Problem of the Week
Problem C
Just Like Kayak 1

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine all the five-digit palindromic integers that are divisible by 55.

NOTE: An integer is divisible by 11 exactly when the alternating sum of its digits is divisible by 11.

To find the alternating sum, start with the first digit and alternate subtracting and adding the remaining digits from left to right. For example, 36 784 is divisible by 11 since $3 - 6 + 7 - 8 + 4 = 0$, and 0 is divisible by 11. However 74 253 is not divisible by 11 since $7 - 4 + 2 - 5 + 3 = 3$, and 3 is not divisible by 11.

Problem of the Week

Problem C and Solution

Just Like Kayak 1

Problem

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Solution

We are looking for all the five-digit integers of the form $abcba$, that are divisible by 55. For an integer to be divisible by 55, it must be divisible by both 11 and 5.

To be divisible by 5, an integer must end in 0 or 5. If a palindrome ends in 0, it must also begin with 0. However, the integer $0bcb0$ is not a five-digit integer since the leading digit is 0. Therefore, the palindromes cannot end with a 0 and hence must start and end with a 5. It follows that any five-digit palindrome divisible by 5 must be of the form $5bcb5$.

For an integer to be divisible by 11, the alternating sum of its digits must be divisible by 11. Therefore, $5 - b + c - b + 5 = 10 - 2b + c$ must divisible by 11. We proceed by looking through the possible values of b.

- If $b = 0$, then $10 - 2(0) + c = 10 + c$ must be divisible by 11. The only solution is $c = 1$. Thus 50 105 is one of the integers.
- If $b = 1$, then $10 - 2(1) + c = 8 + c$ must be divisible by 11. The only solution is $c = 3$. Thus 51 315 is one of the integers.
- If $b = 2$, then $10 - 2(2) + c = 6 + c$ must be divisible by 11. The only solution is $c = 5$. Thus 52 525 is one of the integers.
- If $b = 3$, then $10 - 2(3) + c = 4 + c$ must be divisible by 11. The only solution is $c = 7$. Thus 53 735 is one of the integers.

- If $b = 4$, then $10 - 2(4) + c = 2 + c$ must be divisible by 11. The only solution is $c = 9$. Thus 54 945 is one of the integers.
- If $b = 5$, then $10 - 2(5) + c = c$ must be divisible by 11. The only solution is $c = 0$. Thus 55 055 is one of the integers.
- If $b = 6$, then $10 - 2(6) + c = c - 2$ must be divisible by 11. The only solution is $c = 2$. Thus 56 265 is one of the integers.
- If $b = 7$, then $10 - 2(7) + c = c - 4$ must be divisible by 11. The only solution is $c = 4$. Thus 57 475 is one of the integers.
- If $b = 8$, then $10 - 2(8) + c = c - 6$ must be divisible by 11. The only solution is $c = 6$. Thus 58 685 is one of the integers.
- If $b = 9$, then $10 - 2(9) + c = c - 8$ must be divisible by 11. The only solution is $c = 8$. Thus 59 895 is one of the integers.

Therefore, there are 10 five-digit palindromic integers that are divisible by 55. They are 50 105, 51 315, 52 525, 53 735, 54 945, 55 055, 56 265, 57 475, 58 685, and 59 895.

Problem of the Week
Problem C
Favourite Number 1

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Isaac announced that his favourite number is a five-digit positive integer with a digit sum of 3. Radhika then wrote down a five-digit positive integer with a digit sum of 3. What is the probability that the number Radhika wrote was Isaac's favourite number?

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Favourite Number 1

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2+3+4+5=14$.

Isaac announced that his favourite number is a five-digit positive integer with a digit sum of 3. Radhika then wrote down a five-digit positive integer with a digit sum of 3. What is the probability that the number Radhika wrote was Isaac's favourite number?

Solution

We will first find the groups of five digits that add to 3. Then we will rearrange these digits to determine all five-digit positive integers whose digit sum is 3. Note that the first digit cannot be 0, because otherwise the integer would not be a five-digit integer. This is summarized in the following table.

The five digits	The possible five-digit integers	Number of possibilities
$3,0,0,0,0$	30 000	1
$1,2,0,0,0$	12 000, 21 000, 10 200, 20 100, 10 020, 20 010, 10 002, 20 001	8
$1,1,1,0,0$	11 100, 10 110, 11 010, 10 101, 11 001, 10 011	6

Therefore, the number of five-digit positive integers that have a digit sum of 3 is $1+8+6=15$. Since Radhika wrote down only one of these integers, the probability that this integer was Isaac's favourite number is $\frac{1}{15}$.

Note: It is a known fact that an integer is divisible by 3 exactly when its digit sum is divisible by 3. For example, 32 814 has a digit sum of $3+2+8+1+4=18$. Since 18 is divisible by 3, then 32 814 is divisible by 3. On the other hand, 32 810 has a digit sum of $3+2+8+1+0=14$. Since 14 is not divisible by 3, then 32 810 is not divisible by 3.

As a consequence of this fact, Isaac's favourite number must be divisible by 3.

Problem of the Week

Problem C

Music Ensemble

Three students, Kaila, Octavia, and Sophia, formed a trio for a local music competition and won a cash award. They decided to split the award as follows:

- Kaila received \$100 plus $\frac{1}{5}$ of what then remained;
- Octavia then received \$160 plus $\frac{1}{4}$ of what then remained; and
- Sophia then received the rest, which was \$180.

How much was the original cash award? Which student received the most money?

Theme: Number Sense

Problem of the Week
Problem C and Solution
Music Ensemble

Problem

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- Octavia then received \$160 plus $\frac{1}{4}$ of what then remained; and
- Sophia then received the rest, which was \$180.

How much was the original cash award? Which student received the most money?

Solution

We will start with Sophia, then determine what Octavia received, and then Kaila.

After Kaila got her share and Octavia received \$160, Octavia then received $\frac{1}{4}$ of what remained. Thus, what is left for Sophia is $\frac{3}{4}$ of what remained.
It follows that \$180 is $\frac{3}{4}$ of what remained.

If $\frac{3}{4}$ of the remainder is \$180, then $\frac{1}{4}$ of the remainder is $\$180 \div 3 = \$60$.

Therefore, just after Octavia received \$160, there is $\$60 + \$180 = \$240$ left.
Also, Octavia receives $\$160 + \$60 = \$220$.
Therefore, before Octavia receives any money there is $\$240 + \$160 = \$400$.

Kaila receives $\frac{1}{5}$ of the remainder, so what is left for Octavia is $\frac{4}{5}$ of the remainder.
It follows that \$400 is $\frac{4}{5}$ of the remainder.

If $\frac{4}{5}$ of the remainder is \$400, then $\frac{1}{5}$ of the remainder is $\$400 \div 4 = \$100$.
Therefore, Kaila received $\$100 + \$100 = \$200$.

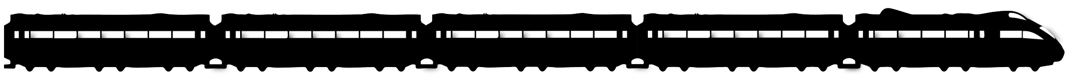
So, just after Kaila received \$100, there is $\$100 + \$400 = \$500$ left.
Therefore, before Kaila receives any money there is $\$500 + \$100 = \$600$.

Therefore, the original monetary award was \$600, and Kaila received \$200, Octavia received \$220, and Sophia received \$180, and so Octavia received the largest share.

We can check our result by working through the given information one step at a time.

Problem of the Week
Problem C
In the Dark

A train that is 400 metres long approaches a tunnel that is 1200 metres long. The entire train is completely in the tunnel for 30 seconds. That is, from the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end of the tunnel, 30 seconds pass. If the train is travelling at a constant speed, determine this speed in kilometres per hour.



Themes: Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
In the Dark

Problem

A train that is 400 metres long approaches a tunnel that is 1200 metres long. The entire train is completely in the tunnel for 30 seconds. That is, from the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end of the tunnel, 30 seconds pass. If the train is travelling at a constant speed, determine this speed in kilometres per hour.

Solution

When the last car of the train has completely entered the tunnel, the front of the train is $1200 - 400 = 800$ metres from the other end of the tunnel. Thus, the train will travel 800 metres in 30 seconds. We can calculate the speed of the train by dividing the distance travelled by the time required to travel the distance.

The speed of the train is $800 \div 30 = \frac{80}{3}$ m/s.

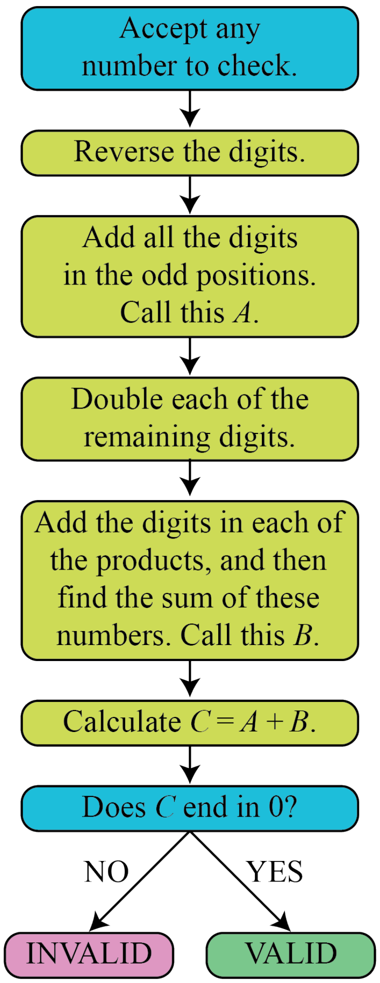
Now we must convert from m/s to km/h. We will do this in two steps: first convert metres to kilometres, and then convert seconds to hours.

1. Metres to kilometres: $\dfrac{80 \text{ m}}{3 \text{ s}} = \dfrac{80 \text{ m}}{3 \text{ s}} \times \dfrac{1 \text{ km}}{1000 \text{ m}} = \dfrac{2 \text{ km}}{75 \text{ s}}$
2. Seconds to hours: $\dfrac{2 \text{ km}}{75 \text{ s}} = \dfrac{2 \text{ km}}{75 \text{ s}} \times \dfrac{60 \text{ s}}{1 \text{ min}} \times \dfrac{60 \text{ min}}{1 \text{ hr}} = \dfrac{96 \text{ km}}{1 \text{ hr}}$

Therefore, the train is travelling at a speed of 96 km/hr.

Problem of the Week
Problem C
Check Please 1

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



Example 1

- Number: 135792
- Reversal: 297531
- Digits in odd positions are underlined: $\underline{2}9\underline{7}5\underline{3}1$
$$\begin{aligned} A &= 2 + 7 + 3 \\ &= 12 \end{aligned}$$
- Double remaining digits:
$$2 \times 9 = 18$$
$$2 \times 5 = 10$$
$$2 \times 1 = 2$$
- Calculate B:
$$\begin{aligned} B &= (1 + 8) + (1 + 0) + 2 \\ &= 9 + 1 + 2 \\ &= 12 \end{aligned}$$
- $C = 12 + 12 = 24$
- C does not end in zero, so the number is not valid.

Example 2

- Number: 1357987
- Reversal: 7897531
- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$
$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$
- Double remaining digits:
$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$
- Calculate B:
$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$
- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number $13X9\ Y659\ 784$ is a valid card number when verified by the Luhn Algorithm, where X and Y are each single digits such that $X \leq Y$.

Determine all possible values of X and Y.

Themes: Algebra, Number Sense

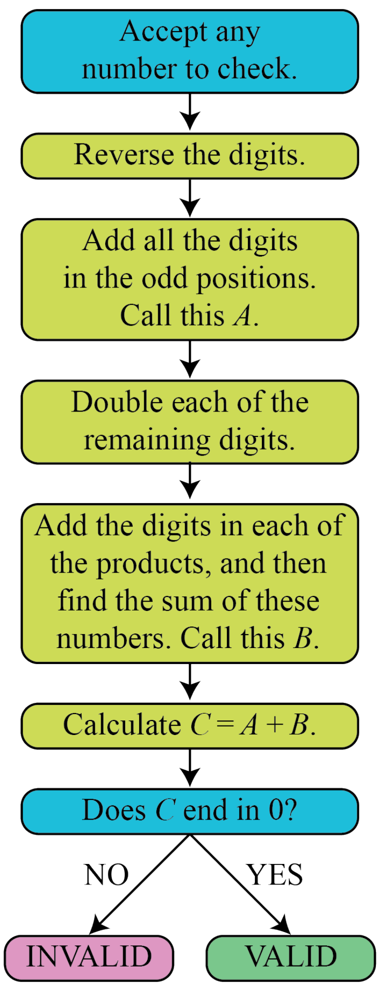
Problem of the Week

Problem C and Solution

Check Please 1

Problem

Debit and credit cards contain account numbers which consist of many digits. When shopping online, customers are often asked to type in their account number. Because there are so many digits, it is easy to make a mistake. The last digit of the number is a specially generated check digit which can be used to verify whether or not the number is valid. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.



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- Double remaining digits:

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- Number: 1357987
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- Digits in odd positions are underlined: $\underline{7}8\underline{9}7\underline{5}3\underline{1}$

$$\begin{aligned} A &= 7 + 9 + 5 + 1 \\ &= 22 \end{aligned}$$

- Double remaining digits:

$$2 \times 8 = 16$$
$$2 \times 7 = 14$$
$$2 \times 3 = 6$$

- Calculate B:

$$\begin{aligned} B &= (1 + 6) + (1 + 4) + 6 \\ &= 7 + 5 + 6 \\ &= 18 \end{aligned}$$

- $C = 22 + 18 = 40$
- C ends in zero, so the number is valid.

The number $13X9\ Y659\ 784$ is a valid card number when verified by the Luhn Algorithm, where X and Y are each single digits such that $X \leq Y$.

Determine all possible values of X and Y.

Solution

When the digits of the card number are reversed the resulting number is $487\ 956Y\ 9X31$. The sum of the digits in the odd positions is $A = 4 + 7 + 5 + Y + X + 1 = 17 + X + Y$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$8 \times 2 = 16;\ 9 \times 2 = 18;\ 6 \times 2 = 12;\ 9 \times 2 = 18;\ 3 \times 2 = 6$$

When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + (1 + 8) + (1 + 2) + (1 + 8) + 6 = 7 + 9 + 3 + 9 + 6 = 34$$

Since $C = A + B$, we have $C = 17 + X + Y + 34 = 51 + X + Y$.

For the card to be valid, the units digit of C must be zero. Since X and Y are digits, the minimum value of $X + Y$ is 0 and the maximum is $9 + 9 = 18$. So C must be between 51 and $51 + 18 = 69$, inclusive. The only value in this range that has a units digit of 0 is 60. Thus, $C = 60$.

Since $C = 51 + X + Y = 60$, it follows that $X + Y = 9$. Since $X \leq Y$, the only possibilities for X and Y are shown in the following table:

X	Y
0	9
1	8
2	7
3	6
4	5

The corresponding card numbers are 1309 9659 784, 1319 8659 784, 1329 7659 784, 1339 6659 784, and 1349 5659 784, which are indeed verified by the Luhn Algorithm.

Problem of the Week
Problem C
Product Puzzle

Three cards are on a table. Each card has a letter on one side and a positive number on the other side. One card has an R on it, one card has a G on it, and one card has a B on it. The number side of each card is face down on the table. The following facts are known about the three concealed numbers:

(i) The product of the number on the card with an R and the number on the card with a G is equal to the number on the card with a B.

(ii) The product of the number on the card with a G and the number on the card with a B is 180.

(iii) Five times the number on the card with a B is equal to the number on the card with a G.

Determine the product of the numbers on the three cards.

Themes: Algebra, Number Sense

Problem of the Week
Problem C and Solution
Product Puzzle

Problem

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(iii) Five times the number on the card with a B is equal to the number on the card with a G.

Determine the product of the numbers on the three cards.

Solution

Solution 1

Let the three numbers be represented by r, g, and b.

Since the product of the number on the card with an R and the number on the card with a G is equal to the number on the card with a B, $r \times g = b$. We are looking for $r \times g \times b = (r \times g) \times b = (b) \times b = b^2$. So when we find b^2 we have found the required product $r \times g \times b$.

We are also given that $g \times b = 180$ and $g = 5 \times b$. Substituting $g = 5 \times b$ into $g \times b = 180$, we have $(5 \times b) \times b = 180$ or $5 \times b^2 = 180$. Dividing by 5, we obtain $b^2 = 36$. This is exactly what we are looking for since $r \times g \times b = b^2$.

Therefore, the product of the three numbers is 36.

For those who would like to know the values of the three numbers, we can continue to solve. We know $b^2 = 36$, so $b = 6$, since b is a positive number. Therefore, $g = 5 \times b = 5 \times 6 = 30$. And finally, $r \times g = b$ so $r \times (30) = 6$. Dividing by 30, we get $r = \frac{6}{30} = \frac{1}{5}$.

We can verify the product $r \times g \times b = (\frac{1}{5}) \times (30) \times (6) = 6 \times 6 = 36$.

Solution 2

In this solution we try to determine the numbers by working with the factors of 180.

The product of the number on the card with a G and the number on the card with a B is 180, and the number on the card with a G is five times the number on the card with a B. The number 180 can be written as $2 \times 2 \times 3 \times 3 \times 5$. If G and B are integers, then G and B must share the same factors, along with G having one more factor, that being 5. By playing with the factors of 180, we find a possibility is that the number on the card with a G is $5 \times 2 \times 3$ and the number on the card with a B is 2×3. That is, the number on the card with a G could be 30 and the number on the card with a B could be 6.

Now using the fact that the number on the card with an R multiplied by the number on the card with a G is equal to the number on the card with a B, we determine that some number multiplied by 30 is equal to 6. It follows that the number on the card with an R would be $6 \div 30 = \frac{1}{5}$.

The product of the three numbers is $\frac{1}{5} \times 30 \times 6 = 6 \times 6 = 36$.

This solution works because the number on the card with a G and the number on the card with a B happen to be integers.

Problem of the Week

Problem C

Soccer Practice

Chris and Pat are practicing soccer. Standing 1 m apart, Pat first kicks the ball to Chris and then Chris kicks the ball back to Pat. Next, standing 2 m apart, Pat kicks the ball to Chris and Chris kicks it back to Pat. After each pair of kicks, Chris moves 1 m farther away from Pat. If they stop playing after the 29^{th} kick, how far apart are they standing and how far did the ball travel in total?

Themes: Algebra, Number Sense

Problem of the Week

Problem C and Solution

Soccer Practice

Problem

Chris and Pat are practicing soccer. Standing 1 m apart, Pat first kicks the ball to Chris and then Chris kicks the ball back to Pat. Next, standing 2 m apart, Pat kicks the ball to Chris and Chris kicks it back to Pat. After each pair of kicks, Chris moves 1 m farther away from Pat. If they stop playing after the 29^{th} kick, how far apart are they standing and how far did the ball travel in total?

Solution

At each distance, two kicks are made: the 1^{st} and 2^{nd} kicks are made when the pair is 1 m apart, the 3^{rd} and 4^{th} kicks are made when they are 2 m apart, and so on, with the 27^{th} and 28^{th} kicks being made when they are 14 m apart. Therefore, the 29^{th} kick is the first kick made when they are 15 m apart. At each distance, the first kick is made by Pat to Chris, so Pat does the 29^{th} kick at a distance of 15 m.

The total distance travelled by the ball is equal to

$$\begin{aligned}
&1+1+2+2+3+3+\cdots+14+14+15 \\
&= (1+2+3+\cdots+14)+(1+2+3+\cdots+14+15) \\
&= \frac{14\times 15}{2}+\frac{15\times 16}{2} \\
&= 105+120 \\
&= 225
\end{aligned}$$

Therefore, the ball travelled a total distance of 225 m.

NOTE:
Did you know that the sum of the first n positive integers is equal to $\frac{n\times(n+1)}{2}$?
That is,

$$1+2+3+\cdots+n=\frac{n\times(n+1)}{2}$$

For example, $1+2+3+4+5=15$, and $\dfrac{5\times 6}{2}=15$.

Also, $1+2+3+4+5+6+7+8=36$, and $\dfrac{8\times 9}{2}=36$.

Problem of the Week
Problem C
Choir Sections

There are 30 students in the school choir at Pascal Middle School. Their conductor split them into two singing groups: altos and sopranos. In total there are 13 altos and 17 sopranos in the choir.

One day there were some students absent from choir practice. Their conductor noticed that more than half the students in the choir were present, and two-thirds of the students present were altos. How many students were absent from choir practice that day?

Themes: Data Management, Number Sense

Problem of the Week
Problem C and Solution
Choir Sections

Problem

There are 30 students in the school choir at Pascal Middle School. Their conductor split them into two singing groups: altos and sopranos. In total there are 13 altos and 17 sopranos in the choir.

One day there were some students absent from choir practice. Their conductor noticed that more than half the students in the choir were present, and two-thirds of the students present were altos. How many students were absent from choir practice that day?

Solution

Solution 1

Since more than half the students in the choir were present that day, that means there were at least 16 students present. Since there were some students absent, that means there were at most 29 students present. Since two-thirds of the students present were altos, that tells us the number of students present must be a multiple of 3. Thus, the number of students present is a multiple of 3 between 16 and 29. The possibilities are 18, 21, 24, and 27.

If there were 18 students present, then $\frac{2}{3} \times 18 = 12$ of them were altos and $18 - 12 = 6$ of them were sopranos. This is a possibility since there are 13 altos and 17 sopranos in the choir.

If there were 21 students present, then $\frac{2}{3} \times 21 = 14$ of them were altos and $21 - 14 = 7$ of them were sopranos. However there are only 13 altos in the choir, so it's not possible that 14 of the students were altos. Thus, this is not a possibility.

If there were more than 21 students present, then the number of altos would be more than 14. Thus, 24 and 27 are also not possibilities for the number of students present.

Therefore, there were 18 students present that day, so $30 - 18 = 12$ students were absent.

Solution 2

This solution uses algebra to solve the problem. Note that the algebra used may be beyond what students have done so far in their math classes.

Let a be the number of altos absent and let s be the number of sopranos absent. Since some students were absent, but more than half the class was present, it follows that the total number of students absent is greater than 0 but less than 15. Thus, $a + s > 0$ and $a + s < 15$.

The number of altos present that day was $13 - a$ and the number of sopranos present was $17 - s$. Since two-thirds of the students present were altos, it follows that one-third of the students present were sopranos. So there were twice as many altos as sopranos. Thus,

$$\begin{aligned} 13 - a &= 2(17 - s) \\ 13 - a &= 34 - 2s \\ 0 &= 21 + a - 2s \\ a &= 2s - 21 \end{aligned}$$

Since a represents the number of altos absent, it follows that $a \geq 0$. So the smallest possible value for s is 11, since $2(10) - 21 = 20 - 21 = -1 < 0$ but $2(11) - 21 = 22 - 21 = 1$. Thus, if $s = 11$ then $a = 1$. In this case, the total number of students absent would be $11 + 1 = 12$.

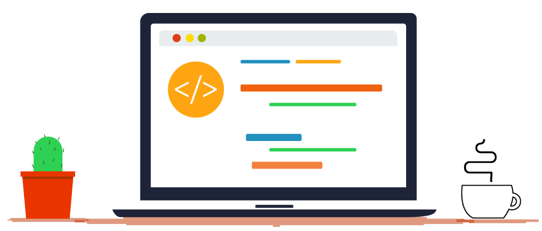
If $s = 12$, then $a = 2(12) - 21 = 24 - 21 = 3$. Then the total number of students absent would be $12 + 3 = 15$. However this is too many, since the number of students absent must be less than 15. If s is any other number larger than 12 then there will be too many students absent. Thus $s = 11$, $a = 1$, and the total number of students absent that day was 12.

Problem of the Week
Problem C
Just a Little Output

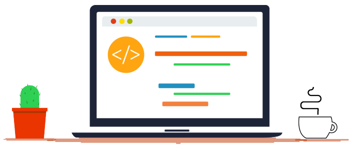
Harlow has written a program. After she inputs a number into the program the number is multiplied by 2, then the product is divided by 3, then 5 is subtracted from the quotient. The final result is then printed.

For example, if Harlow inputs the number 60 into the program then following the steps outlined, the number 60 is first multiplied by 2 to obtain $60 \times 2 = 120$. Then the product is divided by 3 to obtain $120 \div 3 = 40$. Finally, 5 is subtracted from the quotient to obtain $40 - 5 = 35$. The number 35 is then printed.

If Harlow inputs only positive integers into the program, determine the smallest possible positive output number that can be printed.



Themes: Algebra, Number Sense



Problem of the Week

Problem C and Solution

Just a Little Output

Problem

Harlow has written a program. After she inputs a number into the program the number is multiplied by 2, then the product is divided by 3, then 5 is subtracted from the quotient. The final result is then printed.

For example, if Harlow inputs the number 60 into the program then following the steps outlined, the number 60 is first multiplied by 2 to obtain $60 \times 2 = 120$. Then the product is divided by 3 to obtain $120 \div 3 = 40$. Finally, 5 is subtracted from the quotient to obtain $40 - 5 = 35$. The number 35 is then printed.

If Harlow inputs only positive integers into the program, determine the smallest possible positive output number that can be printed.

Solution

We can start by trying different input numbers, starting with 1 and increasing by 1 each time, to see if the output numbers form a pattern. The results for inputs from 1 to 5 are in the following table.

Input	After Multiplying by 2	After Dividing by 3	After Subtracting 5
1	$1 \times 2 = 2$	$2 \div 3 = \frac{2}{3}$	$\frac{2}{3} - \frac{15}{3} = -\frac{13}{3}$
2	$2 \times 2 = 4$	$4 \div 3 = \frac{4}{3}$	$\frac{4}{3} - \frac{15}{3} = -\frac{11}{3}$
3	$3 \times 2 = 6$	$6 \div 3 = 2$	$2 - 5 = -3$
4	$4 \times 2 = 8$	$8 \div 3 = \frac{8}{3}$	$\frac{8}{3} - \frac{15}{3} = -\frac{7}{3}$
5	$5 \times 2 = 10$	$10 \div 3 = \frac{10}{3}$	$\frac{10}{3} - \frac{15}{3} = -\frac{5}{3}$

From this table we can see that when the input numbers are positive integers starting at 1 and increasing by 1 each time, the output numbers form a linear sequence that increases by $\frac{2}{3}$ each time. We now proceed with two different solutions.

Solution 1

In this solution we continue the table until we get the first positive output value.

Input	After Multiplying by 2	After Dividing by 3	After Subtracting 5
6	$6 \times 2 = 12$	$12 \div 3 = 4$	$4 - 5 = -1$
7	$7 \times 2 = 14$	$14 \div 3 = \frac{14}{3}$	$\frac{14}{3} - \frac{15}{3} = -\frac{1}{3}$
8	$8 \times 2 = 16$	$16 \div 3 = \frac{16}{3}$	$\frac{16}{3} - \frac{15}{3} = \frac{1}{3}$

From the table, the first positive output number is $\frac{1}{3}$, and happens when the input number is 8. Since the output numbers form an increasing sequence, we know that $\frac{1}{3}$ is also the smallest possible positive output number.

Note that this solution worked well here because the input number for the smallest possible positive output number was fairly small. If the input number had been much larger than 8 then this wouldn't have been an efficient solution.

Solution 2

This solution uses algebra to solve the problem. Let n be the input number. Following the steps in the program, we first multiply by 2 to obtain $2 \times n$. We then divide by 3 to obtain $\frac{2\times n}{3}$. Finally, we subtract 5 from this quotient to obtain $\frac{2\times n}{3} - 5$. Thus, when the input is n, the output is $\frac{2\times n}{3} - 5$. Since we want the output number to be positive, then

$$\begin{aligned}
\frac{2 \times n}{3} - 5 &> 0 \\
\frac{2 \times n}{3} &> 5 \\
2 \times n &> 5 \times 3 \\
2 \times n &> 15 \\
n &> 15 \div 2 \\
n &> 7.5
\end{aligned}$$

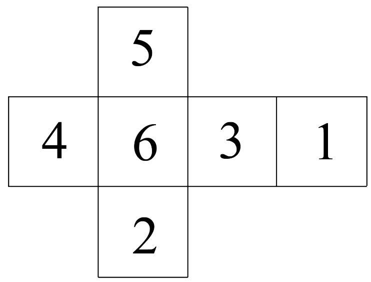
Thus, the input number must be greater than 7.5. Since the input number must be an integer, it follows that the smallest input number with a positive output number is 8. Since the output numbers form an increasing sequence, an input of 8 will give the smallest possible positive output number. When the input number is 8, the output number is:

$$\frac{2 \times 8}{3} - 5 = \frac{16}{3} - 5 = \frac{16}{3} - \frac{15}{3} = \frac{1}{3}$$

Therefore, if the input number is a positive integer, then the smallest possible positive output number is $\frac{1}{3}$.

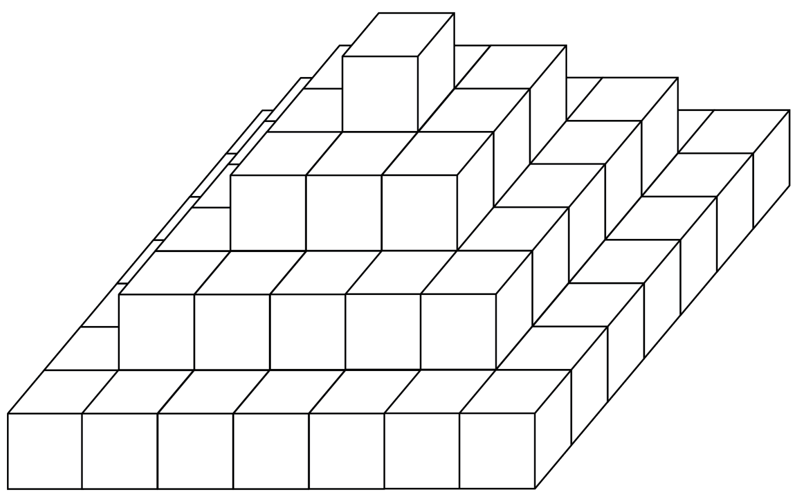
Problem of the Week
Problem C
Counting Numbered Cubes

Dmitri has a collection of identical cubes. Each cube is labelled with the integers 1 to 6, as shown in the following net:



(This net can be folded to make a cube.)

He forms a pyramid by stacking layers of the cubes on a table, as shown, with the bottom layer being a 7 by 7 square of cubes.



(a) Determine the total number of cubes used to build the pyramid.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) He wants to position the cubes so that when all of the visible numbers are added up, the total is as large as possible. What is this total?

Themes: Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Counting Numbered Cubes

Problem

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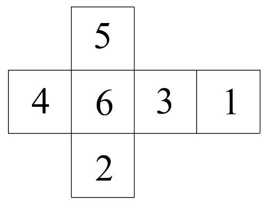

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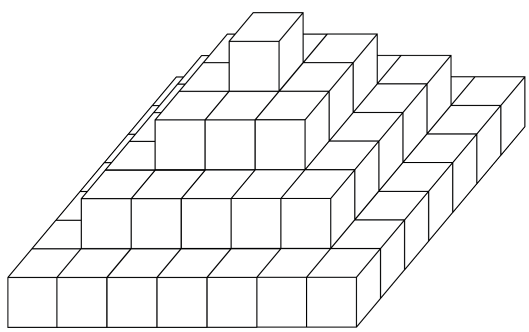

(a) Determine the total number of cubes used to build the pyramid.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) He wants to position the cubes so that when all of the visible numbers are added up, the total is as large as possible. What is this total?

Solution

(a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses $7 \times 7 = 49$ cubes.
The next layer of the cube is a 5 by 5 square of cubes, so uses $5 \times 5 = 25$ cubes.
The next layer of the cube is a 3 by 3 square of cubes, so uses $3 \times 3 = 9$ cubes.
The top layer consists of a single cube.
Therefore, the total number of cubes used is $49 + 25 + 9 + 1 = 84$.

(b) **Solution 1**

The cube in the top layer has 5 visible faces (only the bottom face is hidden).

In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).

In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 3 \times 2 = 24$ visible faces).

In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 5 \times 2 = 40$ visible faces).

Therefore, the total number of visible faces is
$5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$.

Solution 2

When we look at the pyramid from the top, we see a 7 by 7 square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)

When we look at the pyramid from each of the four sides, we see
$1 + 3 + 5 + 7 = 16$ visible faces, so there are $4(16) = 64$ visible faces on the sides.

Therefore, there are $49 + 64 = 113$ visible faces in total.

(c) **Solution 1**

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the "1" on the bottom face (and so is hidden). The total of the numbers visible on this cube is $2 + 3 + 4 + 5 + 6 = 20$.

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5, and 6 all visible (this is possible since the faces with the 4, 5, and 6 share a vertex) and the 1, 2, and 3 hidden.

There are 12 of these cubes, so the total of the numbers visible on these cubes is $12 \times (4 + 5 + 6) = 180$.

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.

There are $4 + 12 + 20 = 36$ of these cubes, so the total of the numbers visible on these cubes is $36 \times (5 + 6) = 396$.

Therefore, the overall largest possible total is $20 + 180 + 396 = 596$.

Solution 2

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top face which is visible is 6 (for a total of $49 \times 6 = 294$).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds $2 + 3 + 4 + 5 = 14$ to the total of the visible faces thus far.

Consider lastly the faces visible on the sides (a total of $4 \times (3 + 5 + 7) = 60$ faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s).

This would give a total of $5 \times 60 = 300$.

However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5, and 6 meet at a vertex on each cube.)

Therefore, the overall largest possible total is $294 + 14 + 300 - 12 = 596$.