Problem of the Week
Problem C and Solution
Just Like Kayak 1

Problem

A palindrome is a word, phrase, or positive integer that reads the same forwards and backwards. For example, "kayak" is a palindrome. The integers 292, 11, and 6 357 536 are also palindromes.

Determine all the five-digit palindromic integers that are divisible by 55.

NOTE: An integer is divisible by 11 exactly when the alternating sum of its digits is divisible by 11.

To find the alternating sum, start with the first digit and alternate subtracting and adding the remaining digits from left to right. For example, 36 784 is divisible by 11 since $3 - 6 + 7 - 8 + 4 = 0$, and 0 is divisible by 11. However 74 253 is not divisible by 11 since $7 - 4 + 2 - 5 + 3 = 3$, and 3 is not divisible by 11.

Solution

We are looking for all the five-digit integers of the form $abcba$, that are divisible by 55. For an integer to be divisible by 55, it must be divisible by both 11 and 5.

To be divisible by 5, an integer must end in 0 or 5. If a palindrome ends in 0, it must also begin with 0. However, the integer $0bcb0$ is not a five-digit integer since the leading digit is 0. Therefore, the palindromes cannot end with a 0 and hence must start and end with a 5. It follows that any five-digit palindrome divisible by 5 must be of the form $5bcb5$.

For an integer to be divisible by 11, the alternating sum of its digits must be divisible by 11. Therefore, $5 - b + c - b + 5 = 10 - 2b + c$ must divisible by 11. We proceed by looking through the possible values of b.

- If $b = 0$, then $10 - 2(0) + c = 10 + c$ must be divisible by 11. The only solution is $c = 1$. Thus 50 105 is one of the integers.
- If $b = 1$, then $10 - 2(1) + c = 8 + c$ must be divisible by 11. The only solution is $c = 3$. Thus 51 315 is one of the integers.
- If $b = 2$, then $10 - 2(2) + c = 6 + c$ must be divisible by 11. The only solution is $c = 5$. Thus 52 525 is one of the integers.
- If $b = 3$, then $10 - 2(3) + c = 4 + c$ must be divisible by 11. The only solution is $c = 7$. Thus 53 735 is one of the integers.

- If $b = 4$, then $10 - 2(4) + c = 2 + c$ must be divisible by 11. The only solution is $c = 9$. Thus 54 945 is one of the integers.
- If $b = 5$, then $10 - 2(5) + c = c$ must be divisible by 11. The only solution is $c = 0$. Thus 55 055 is one of the integers.
- If $b = 6$, then $10 - 2(6) + c = c - 2$ must be divisible by 11. The only solution is $c = 2$. Thus 56 265 is one of the integers.
- If $b = 7$, then $10 - 2(7) + c = c - 4$ must be divisible by 11. The only solution is $c = 4$. Thus 57 475 is one of the integers.
- If $b = 8$, then $10 - 2(8) + c = c - 6$ must be divisible by 11. The only solution is $c = 6$. Thus 58 685 is one of the integers.
- If $b = 9$, then $10 - 2(9) + c = c - 8$ must be divisible by 11. The only solution is $c = 8$. Thus 59 895 is one of the integers.

Therefore, there are 10 five-digit palindromic integers that are divisible by 55. They are 50 105, 51 315, 52 525, 53 735, 54 945, 55 055, 56 265, 57 475, 58 685, and 59 895.