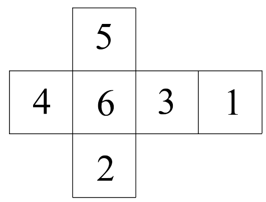
Problem of the Week
Problem C and Solution
Counting Numbered Cubes

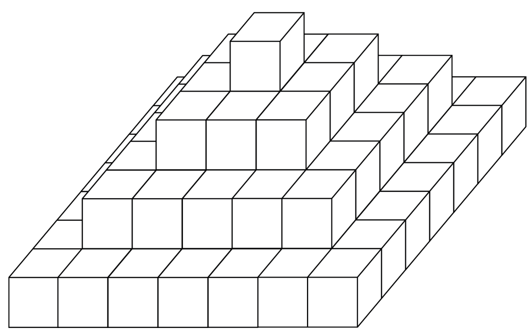
Problem

Dmitri has a collection of identical cubes. Each cube is labelled with the integers 1 to 6, as shown in the following net:



(This net can be folded to make a cube.)

He forms a pyramid by stacking layers of the cubes on a table, as shown, with the bottom layer being a 7 by 7 square of cubes.



(a) Determine the total number of cubes used to build the pyramid.

(b) How many faces are visible after the pyramid is built and sitting on the table?

(c) He wants to position the cubes so that when all of the visible numbers are added up, the total is as large as possible. What is this total?

Solution

(a) The bottom layer of the cube is a 7 by 7 square of cubes, so uses $7 \times 7 = 49$ cubes.
The next layer of the cube is a 5 by 5 square of cubes, so uses $5 \times 5 = 25$ cubes.
The next layer of the cube is a 3 by 3 square of cubes, so uses $3 \times 3 = 9$ cubes.
The top layer consists of a single cube.
Therefore, the total number of cubes used is $49 + 25 + 9 + 1 = 84$.

(b) **Solution 1**

The cube in the top layer has 5 visible faces (only the bottom face is hidden).

In the second layer from the top, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there is 1 cube on each of the four sides of the layer with 2 visible faces (another 8 visible faces).

In the second layer from the bottom, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 3 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 3 \times 2 = 24$ visible faces).

In the bottom layer, the 4 corner cubes each have 3 visible faces (for 12 faces in total), and there are 5 cubes on each of the four sides of the layer with 2 visible faces (another $4 \times 5 \times 2 = 40$ visible faces).

Therefore, the total number of visible faces is
$5 + 12 + 8 + 12 + 24 + 12 + 40 = 113$.

Solution 2

When we look at the pyramid from the top, we see a 7 by 7 square of visible faces, or 49 visible faces. (This square is composed of faces from all of the levels of the pyramid.)

When we look at the pyramid from each of the four sides, we see
$1 + 3 + 5 + 7 = 16$ visible faces, so there are $4(16) = 64$ visible faces on the sides.

Therefore, there are $49 + 64 = 113$ visible faces in total.

(c) **Solution 1**

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

For the top cube (with 5 visible faces), we position this cube with the "1" on the bottom face (and so is hidden). The total of the numbers visible on this cube is $2 + 3 + 4 + 5 + 6 = 20$.

For the 4 corner cubes on each layer (each with 3 visible faces), we position these cubes with the 4, 5, and 6 all visible (this is possible since the faces with the 4, 5, and 6 share a vertex) and the 1, 2, and 3 hidden.

There are 12 of these cubes, so the total of the numbers visible on these cubes is $12 \times (4 + 5 + 6) = 180$.

For the cubes on the sides (that is, not at the corner) of each layer, we position the cubes with the 5 and 6 visible (this is possible since the faces with 5 and 6 share an edge) and the 1, 2, 3, and 4 hidden.

There are $4 + 12 + 20 = 36$ of these cubes, so the total of the numbers visible on these cubes is $36 \times (5 + 6) = 396$.

Therefore, the overall largest possible total is $20 + 180 + 396 = 596$.

Solution 2

To make the total of all of the visible numbers as large as possible, we should position the cubes so that the largest possible two, three, or five numbers are visible, depending on its position.

As in Solution 2 to (b), view the pyramid from the top. Position the cubes so that each top face which is visible is 6 (for a total of $49 \times 6 = 294$).

Consider next the top cube. It has only one hidden face, which will be the 1, since the 6 is on top. (This does maximize the sum of the visible faces.) This adds $2 + 3 + 4 + 5 = 14$ to the total of the visible faces thus far.

Consider lastly the faces visible on the sides (a total of $4 \times (3 + 5 + 7) = 60$ faces). To maximize the sum of the numbers on these faces, we would like to make them all 5 (since we have already used the 6s).

This would give a total of $5 \times 60 = 300$.

However, there are 12 corner cubes to which we have assigned two 5s, so we must change one of the 5s on each to a 4, decreasing the total by 12. (This is possible, since the 4, 5, and 6 meet at a vertex on each cube.)

Therefore, the overall largest possible total is $294 + 14 + 300 - 12 = 596$.