Problem of the Week
Problem C and Solution
Favourite Number 1

Problem

The *digit sum* of a positive integer is the sum of all its digits. For example, the digit sum of the integer 2345 is $2 + 3 + 4 + 5 = 14$.

Isaac announced that his favourite number is a five-digit positive integer with a digit sum of 3. Radhika then wrote down a five-digit positive integer with a digit sum of 3. What is the probability that the number Radhika wrote was Isaac's favourite number?

Solution

We will first find the groups of five digits that add to 3. Then we will rearrange these digits to determine all five-digit positive integers whose digit sum is 3. Note that the first digit cannot be 0, because otherwise the integer would not be a five-digit integer. This is summarized in the following table.

The five digits	The possible five-digit integers	Number of possibilities
3, 0, 0, 0, 0	30 000	1
1, 2, 0, 0, 0	12 000, 21 000, 10 200, 20 100, 10 020, 20 010, 10 002, 20 001	8
1, 1, 1, 0, 0	11 100, 10 110, 11 010, 10 101, 11 001, 10 011	6

Therefore, the number of five-digit positive integers that have a digit sum of 3 is $1 + 8 + 6 = 15$. Since Radhika wrote down only one of these integers, the probability that this integer was Isaac's favourite number is $\frac{1}{15}$.

Note: It is a known fact that an integer is divisible by 3 exactly when its digit sum is divisible by 3. For example, 32 814 has a digit sum of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 3, then 32 814 is divisible by 3. On the other hand, 32 810 has a digit sum of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 3, then 32 810 is not divisible by 3.

As a consequence of this fact, Isaac's favourite number must be divisible by 3.