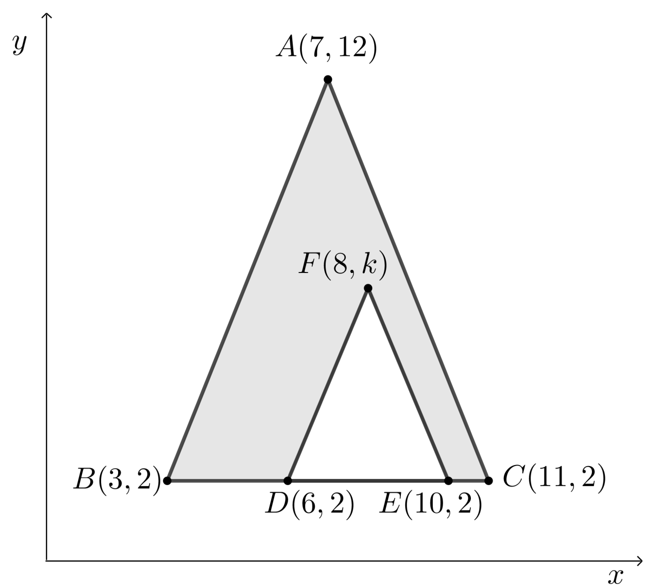
Problem of the Week
Problem C and Solution
Tangled Triangles

Problem

Points $A(7,12)$, $B(3,2)$, $C(11,2)$, $D(6,2)$ and $E(10,2)$ are plotted on the Cartesian plane. The point $F(8,k)$ lies inside $\triangle ABC$ so that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2.

Determine the value of k, the y-coordinate of F.



Solution

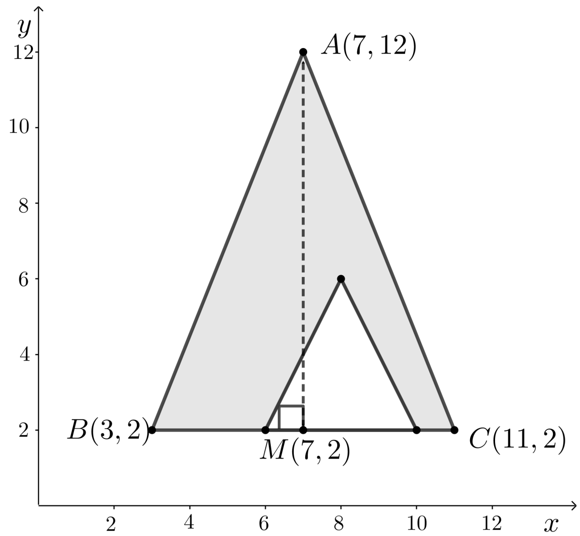
We will calculate the area of $\triangle ABC$, calculate the area of $\triangle DEF$, and then use the given information that the difference between these areas is equal to 32 units^2.

We will use the fact that the distance between two points that have the same x-coordinate is the positive difference between their y-coordinates. We will also use the fact that the distance between two points that have the same y-coordinate is the positive difference between their x-coordinates.

Since B and C both have y-coordinate 2, BC is a horizontal line. Thus, $BC = 11 - 3 = 8$.

In $\triangle ABC$, drop a perpendicular from vertex A to M on BC. Since BC is horizontal, then AM is vertical. Since every point on a vertical line has the same x-coordinate, M has x-coordinate 7. Similarly, since M is on the horizontal line through $B(3,2)$ and $C(11,2)$, M has y-coordinate 2. Therefore, $AM = 12 - 2 = 10$.

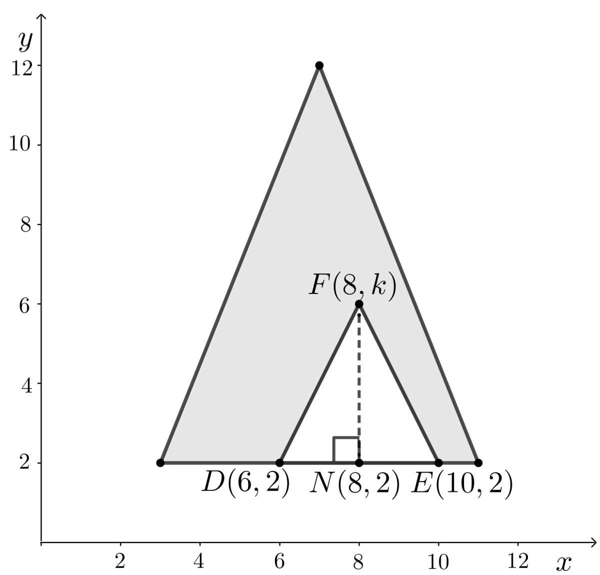
Thus, the area of $\triangle ABC$ is equal to $\dfrac{BC \times AM}{2} = \dfrac{8 \times 10}{2} = 40$.



Since D and E both have y-coordinate 2, DE is a horizontal line. Thus, $DE = 10 - 6 = 4$.

In $\triangle DEF$, drop a perpendicular from vertex F to N on DE. Since DE is horizontal, then FN is vertical. Since every point on a vertical line has the same x-coordinate, N has x-coordinate 8. Similarly, since N is on the horizontal line through $D(6, 2)$ and $E(10, 2)$, N has y-coordinate 2. Therefore, $FN = k - 2$.

Thus, the area of $\triangle DEF$ is equal to $\dfrac{DE \times FN}{2} = \dfrac{4 \times (k-2)}{2} = 2(k-2)$.



We now use the given information that the area inside of $\triangle ABC$ but outside of $\triangle DEF$ is equal to 32 units^2 to obtain the equation $40 - 2(k-2) = 32$.

Subtracting 32 from each side, we have $8 - 2(k-2) = 0$.

Adding $2(k-2)$ to each side, we have $8 = 2(k-2)$, which simplifies to $4 = k - 2$, or $k = 6$.

Therefore, the value of k, the y-coordinate of point F, is 6.