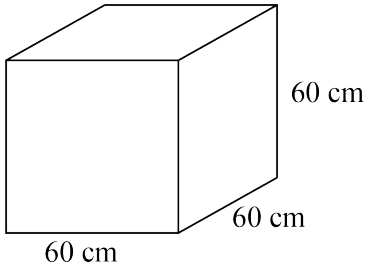
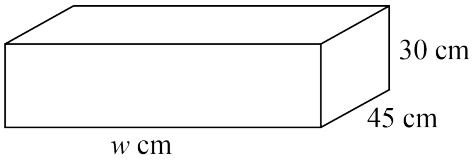
Problem of the Week
Problem C and Solution
Hidden Space

Problem

Two students are building storage boxes in woodworking class. The first student builds a box in the form of a cube with each edge measuring 60 cm. The second student builds a box in the shape of a rectangular prism with width w cm, height 30 cm and depth 45 cm. Surprisingly, each student's box has the same total surface area.

Determine which student's box has the greatest volume. How much greater is the volume?





Solution

To determine the surface area of a cube, we determine the area of one side and multiply by 6. Therefore, the total surface area of the cube-shaped box is $6 \times 60 \times 60 = 21\,600$ cm^2.

To determine the total surface area of the rectangular prism, we determine the sum of the areas of the six sides. The front side and the back side have equal areas. The top and the bottom have equal areas. Each of the remaining two sides have equal area. Therefore, the total surface area of the rectangular prism box is

$$\begin{aligned} 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of side} &= 2 \times 30 \times w + 2 \times 45 \times w + 2 \times 30 \times 45 \\ &= 60w + 90w + 2700 \\ &= 150w + 2700 \end{aligned}$$

But the total surface area of the cube-shaped box is the same as the total surface area of the rectangular prism. Therefore,

$$\begin{aligned} 21\,600 &= 150w + 2700 \\ 21\,600 - 2700 &= 150w + 2700 - 2700 \\ 18\,900 &= 150w \\ \frac{18\,900}{150} &= \frac{150w}{150} \\ 126 &= w \end{aligned}$$

Therefore, the width of the rectangular prism is 126 cm.

To determine the volume of a cube, we "cube" the edge length. So, the volume of the cube is $60 \times 60 \times 60 = 60^3 = 216\,000$ cm^3.

To determine the volume of the rectangular prism, we multiply the three different edge lengths. So, the volume of the rectangular prism is $126 \times 45 \times 30 = 170\,100$ cm^3.

Therefore, the student who is building the cube has the box with the greater volume. The volume of the cube is greater by $216\,000 - 170\,100 = 45\,900$ cm^3.