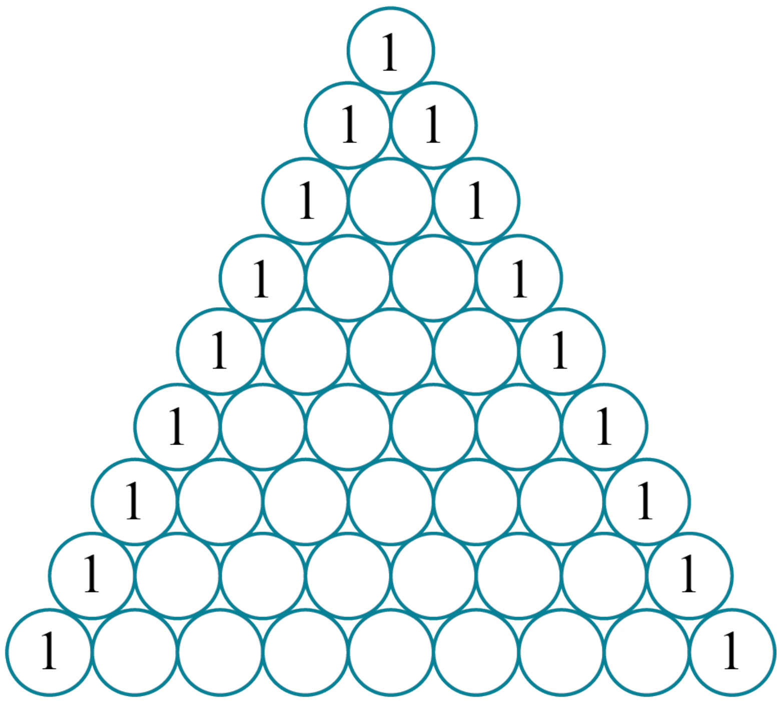
Problem of the Week
Problem A and Solution
Pencil Pyramid

Problem

Juliane arranges identical cylindrical tubes to form a pyramid. She puts one pencil in the top tube and then puts one pencil in the leftmost and rightmost tubes in each of the remaining rows, as shown.



Juliane then fills the remaining tubes as follows:

Step 1: Find the topmost empty tube and add up the number of pencils in the two tubes touching it in the row above.

Step 2: Put that many pencils in the empty tube.

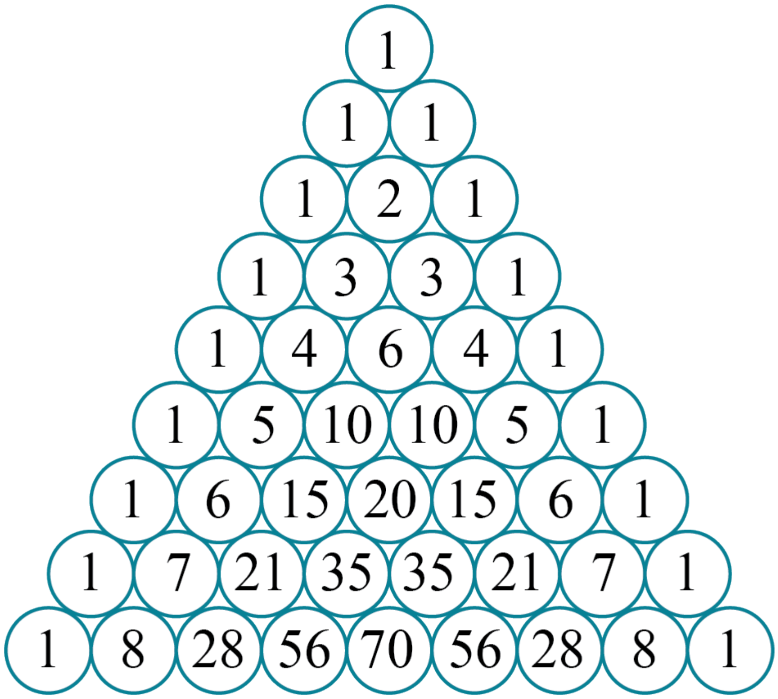
Step 3: Repeat Steps 1 and 2 until all tubes are filled.

(a) Complete the pyramid with the number of pencils in each tube.

(b) Can you find any patterns in the numbers in the completed pyramid? If so, explain them.

Solution

(a) The completed pyramid is shown.



(b) Here are some patterns you may have noticed:

- The triangle is symmetric, so the left side matches the right side.
- Starting with the leftmost 1 in the second row and moving down diagonally we see the numbers 1, 2, 3, 4, 5, 6, 7, and 8.
- Starting with the leftmost 1 in the third row and moving down diagonally we see the numbers 1, 3, 6, 10, 15, 21, and 28. The pattern rule for this sequence is to add 2, then 3, then 4, then 5, etc.
- The sum of the numbers in any row is twice the sum of the numbers in the row above it.
- If you start from any 1, follow the diagonal down, and then change directions for the last number, the sum of the numbers in the diagonal will equal the last number. For example, $1 + 6 + 21 = 28$. This is called the *Hockey Stick Identity* because if you circle the numbers used, it will look somewhat like a hockey stick.

Teacher's Notes

This problem is an exploration of *Pascal's Triangle*, which is named for French mathematician Blaise Pascal (1623 - 1662). There are lots of patterns to be found in this triangle. We listed some of the patterns in the solution, but perhaps your students were able to find others!

Pascal's Triangle is also used in algebra. The numbers in the n^{th} row provide the coefficients for the expansion of a *binomial expression* raised to the n^{th} power. The expansions for $n = 0$ to $n = 4$ are shown.

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= 1x + 1y \\
(x+y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x+y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4
\end{aligned}
$$

As the value of n increases, expanding the binomials manually becomes quite time consuming. Using Pascal's Triangle allows us to expand binomial expressions more efficiently.