



Problem of the Month

Problem 8: Revolutionary counting

May 2026

This Problem of the Month is inspired by Question 24 of the Team Problems from the CEMC's 2026 Canadian Team Mathematics Contest:

In a 4×4 grid consisting of 16 unit squares, 4 squares are coloured red, while the other 12 are coloured white. Two colourings are said to be equivalent if one can be obtained from the other by doing a sequence of 90° rotations. How many inequivalent colourings are there?

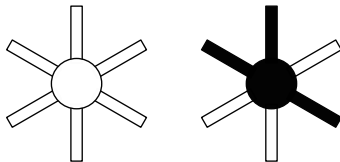
Try the problem as a warm up! The following problems will investigate counting problems of the form “in how many inequivalent ways can you colour something where two colourings are equivalent if one can be obtained from the other by a rotation?”

- Consider a 2×2 grid of four unit squares, where each square is coloured either black or white. Let r_1, r_2, r_3 , and r_0 denote the rotations of the grid by 90° clockwise, 180° clockwise, 270° clockwise, and 0° respectively. View these four rotations as functions that take as an input a colouring of the grid, and give as an output a colouring. For example,

$$r_1 \left(\begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \square & \square \\ \hline \end{array}, \quad r_2 \left(\begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \blacksquare & \square \\ \hline \end{array},$$

and $r_0(c) = c$ for all colourings c . Two colourings c and d are equivalent if there is a rotation r so that $r(c) = d$.

- How many inequivalent colourings of the 2×2 grid exist?
 - For each rotation r , let $\text{fix}(r)$ be the number of colourings c satisfying $r(c) = c$. Compute $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3)$.
 - Compare your answers to the previous two parts.
- Consider a windmill with six identical blades evenly spaced around a central circle. The central circle, and each of the blades, are to be coloured either black or white. Let r_0, r_1, r_2, r_3, r_4 , and r_5 denote the rotations of the windmill by $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ$, and 300° clockwise, respectively. Two colourings c and d are equivalent if there is a rotation r such that $r(c) = d$. The image below shows two inequivalent colourings of the windmill.



- How many inequivalent colourings of the windmill are there?
- For each rotation r let $\text{fix}(r)$ denote the number of colourings c of the windmill satisfying $r(c) = c$. Compute $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3) + \text{fix}(r_4) + \text{fix}(r_5)$.



- (c) Compare your answers to the previous two parts.
3. Now for a more general approach! Suppose we have some object and k rotations r_0, r_1, \dots, r_{k-1} we can perform on the object, where r_z is a rotation by $\frac{360z}{k}$ degrees clockwise. Assume that the object can be coloured according to some rules, and consider the collection of all possible colourings of the object according to the rules. In Question 1 for example, the object is the 2×2 grid, $k = 4$ since there are four rotations, and there are $2^4 = 16$ possible colourings of the grid where each square is either black or white. Two colourings c and d are *equivalent* if there is a rotation r so that $r(c) = d$. Let N be the number of inequivalent colourings of the object.
- (a) Fix a colouring c . Let $E(c)$ be the number of colourings (including c itself) that are equivalent to c . Let $S(c)$ be the number of rotations satisfying $r(c) = c$. Show that $E(c)S(c) = k$.
- (b) A *stable pair* is a pair (r, c) where r is a rotation, c is a colouring, and $r(c) = c$. Show that the number of stable pairs is equal to kN .
- (c) For a rotation r , let $\text{fix}(r)$ denote the number of colourings c satisfying $r(c) = c$. Show that the number of stable pairs is equal to $\text{fix}(r_0) + \text{fix}(r_1) + \dots + \text{fix}(r_{k-1})$.
4. Consider the 4×4 grid of 16 unit squares given at the beginning of this document. Let r_0, r_1, r_2 , and r_3 be the rotations by $0^\circ, 90^\circ, 180^\circ$, and 270° clockwise, respectively. By computing $\text{fix}(r_0) + \text{fix}(r_1) + \text{fix}(r_2) + \text{fix}(r_3)$, solve the problem at the beginning of this document.
5. Let p be a prime. Consider a windmill with p identical blades evenly spaced around a central circle. The central circle, and each of the blades, are to be coloured either black or white. Two colourings are considered equivalent if you can rotate one into the other. How many inequivalent colourings of the windmill are there?
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