Problem of the Month

Solution to Problem 6: Cobbling cards

March 2026

1. Here is such a Cobble deck:

$$[A, B, E], [A, C, F], [A, D, G], [B, C, G], [B, D, F], [C, D, E], [E, F, G].$$

You should check that it satisfies the four properties from the definition given the problem statement. There are many other ways this could be done, but they all differ from this one by permuting the letters and permuting the cards.

To see how you could have constructed the deck, read through the solution to Question 2 including the note at the end.

2. Suppose A, B, C, and D are four symbols satisfying Property (iv). Then Properties (iii) and (iv) together tell us that each pair of these four symbols appears on a distinct card. This is because if two pairs appeared on the same card, we would have three of the four symbols appearing on the same card, contradicting Property (iv).

There are six distinct pairs you can create from the four symbols A, B, C, and D, and each needs to be on a distinct card. So, the six cards must look like

$$K_1 = [A, B, \dots] \quad K_2 = [A, C, \dots] \quad K_3 = [A, D, \dots]$$
$$K_4 = [B, C, \dots] \quad K_5 = [B, D, \dots] \quad K_6 = [C, D, \dots].$$

We have argued that there must be at least six cards in a Cobble deck, so to complete the question we must show that you cannot add symbols to these six cards to create a Cobble deck.

So far, this collection of cards satisfies Properties (i), (iii), and (iv), but not (ii). Currently, K_1 and K_6 do not have a symbol in common, K_2 and K_5 do not have a symbol in common, and K_3 and K_4 do not have a symbol in common.

Let's first look at K_1 and K_6. We cannot add C or D to K_1, since then A, B, C, and D would no longer satisfy Property (iv). Similarly, we cannot add A or B to K_6. Therefore, we need to introduce a new symbol to both K_1 and K_6. Call this new symbol E. The six cards must therefore look like this:

$$K_1 = [A, B, E, \dots] \quad K_2 = [A, C, \dots] \quad K_3 = [A, D, \dots]$$
$$K_4 = [B, C, \dots] \quad K_5 = [B, D, \dots] \quad K_6 = [C, D, E, \dots].$$

Let's now focus on K_2 and K_5. By a similar argument as we made for K_1 and K_6, we cannot add A, B, C, or D to either K_2 or K_5 to make the two cards contain a unique symbol in common. Furthermore, we cannot add E to both K_2 and K_5 because then K_2 and K_1 would share more than one symbol (and so would K_5 and K_6, K_2 and K_6, and K_1 and K_5).

We are therefore forced to introduce another symbol F to both K_2 and K_5.

A similar argument can be made for K_3 and K_4, whose conclusion is that yet another symbol G must be introduced to both cards.

Therefore, assuming these six cards form a Cobble deck, they must look like this:

$$K_1 = [A, B, E, \ldots] \quad K_2 = [A, C, F, \ldots] \quad K_3 = [A, D, G, \ldots]$$
$$K_4 = [B, C, G, \ldots] \quad K_5 = [B, D, F, \ldots] \quad K_6 = [C, D, E, \ldots].$$

These six cards now satisfy Properties (i), (ii), and (iv). However, (iii) is not satisfied since E and F (for example) do not appear together on any card.

The goal now is to force E and F to appear together on a unique card. Let's try to do this for each card one by one.

K_1: Introducing an F to K_1 would mean K_1 and K_2 share more than one symbol, so we cannot do this.

K_2: Adding E to K_2 means K_1 and K_2 share two symbols, an unacceptable state of affairs.

K_3: Putting both E and F on K_3 would force K_2 and K_3 to share A and F, which rules this possibility out.

K_4: Like the previous case, adding E and F to K_4 implies K_4 and K_5 share two symbols.

K_5: Placing E on K_5 is also not possible since K_5 and K_6 would then share two symbols.

K_6: Adding an F to K_6 forces K_5 and K_6 to have two symbols in common, which is simply not an option.

There is no path forward without introducing another card, and therefore there is no Cobble deck with six cards.

Note: We can remedy the situation by introducing a seventh card $[E, F, G]$, which results in the Cobble deck from the solution to Question 1.

3. We'll first prove a helpful result (which we will call a lemma, a word used in mathematics for an intermediary result used to prove a more desirable result).

Lemma: For any two cards in a Cobble deck, there is a symbol that is not on either card.

Proof. This proof will be a little sneaky. Our strategy will be to assume there are two cards that contain all the symbols that appear in the deck, and arrive at a contradiction.

Assume that K_1 and K_2 contain all the symbols in the Cobble deck. By Property (iv), there are four symbols so that no three of them appear on the same card. Therefore, two of them must appear on K_1 and two must appear on K_2. Let A and B be the two symbols appearing on K_1, and let C and D be the two symbols appearing on K_2. Note that none of A, B, C, or D appear on both K_1 and K_2 since then three of them would appear on one of the cards.

Let L_1 be the unique card containing A and C, and let L_2 be the unique card containing B and D. Note that L_1, L_2, K_1, and K_2 must be four distinct cards.

Let E be the unique symbol appearing on both L_1 and L_2. Note that E is distinct from A and B since A is not on L_2 and B is not on L_1 (can you see why?). The goal is to show that E cannot appear on K_1 or K_2, which will contradict our original assumption and complete the proof.

If E is on K_1, then E and A are both on L_1 and K_1, which cannot happen. If E is on K_2, then E and B are both on L_2 and K_2, which is impossible. Therefore, E is not on K_1 or K_2, completing the proof. □

Excellent! Let's now use the Lemma to give a solution to the original question. Let K_1 and K_2 be two distinct cards. Let P be a symbol not contained in either K_1 and K_2, which much exist by the Lemma. Suppose $K_1 = [A_1, A_2, \ldots, A_n]$. For each i between 1 and n, let L_i be the unique card containing A_i and P. Let B_i be the unique symbol contained in both L_i and K_2.

We want to show that $K_2 = [B_1, B_2, \ldots, B_n]$. The first thing is to show that each of the B_i are distinct. Suppose not, and suppose that $B_i = B_j$ for some $i \neq j$. Then B_i would be on both L_i and L_j. Since P is the unique symbol on both L_i and L_j, this would mean that $B_i = P$. However, P is not on K_2, a contradiction. Therefore, the symbols $B_1, B_2, \ldots, B_n$ are all distinct.

We must now show that there are no other symbols on K_2. To that end, suppose B is a symbol on K_2, and let L be the unique card containing B and P. Now, L and K_1 share a symbol. Since $K_1 = [A_1, A_2, \ldots, A_n]$, it must be the case that there is a unique t between 1 and n so that A_t is on both L and K_1. Since L contains A_t and P, it must be that $L = L_t$. Since B_t is the unique symbol on both L_t and K_2, and since B is contained on both L and K_2, we are forced to conclude that $B = B_t$.

We now have that $K_2 = [B_1, B_2, \ldots, B_n]$. Therefore, if a card contains n symbols, then every other card must also contain n symbols.

4. One of the (many) amazing things about Cobble decks is that roughly, you can change the roles of cards and symbols and prove similar statements. For example, the result we will prove in this question is the counterpart to the one in the previous question.

   To begin this question, we will first prove the counterpart of Property (iv), which we will call Property (v).

   Property (v): In a Cobble deck, there are four cards so that no three of them contain the same symbol.

   Proof. Let A, B, C, and D be the four symbols satisfying Property (iv) for the Cobble deck. Let K_1, K_2, K_3, and K_4 be the unique cards that contain A and B, B and C, C and D, and D and A respectively. The claim is that these four cards satisfy the property that no three of them contain the same symbol.

   The unique symbol common to K_1 and K_2 is B, the unique symbol common to K_2 and K_3 is C, the unique symbol common to K_3 and K_4 is D, and the unique symbol common to K_4 and K_1 is A.

   Since K_2 contains C, K_1 contains A, and the unique symbol common to K_1 and K_2 is B, K_1 does not contain C and K_2 does not contain A. Similarly,

   - K_2 does not contain D and K_3 does not contain B,

- K_3 does not contain A and K_4 does not contain C, and
- K_4 does not contain B and K_1 does not contain D.

Since K_3 does not contain the unique symbol common to K_1 and K_2 (which is B), there is no symbol that appears on K_1, K_2, and K_3. Since K_4 also does not contain B, there is no symbol appearing on the three cards K_1, K_2, and K_4.

Similarly, K_1 and K_2 do not contain the unique symbol common to K_3 and K_4 (which is D), so there is no symbol that appears on all three cards K_1, K_3, K_4, as well as K_2, K_3, K_4. This checks every possibility of three cards from the set K_1, K_2, K_3, K_4, and these four cards satisfy that no three of them contain the same symbol. □

Now, Properties (ii) and (iii) are counterparts of each other, and Properties (iv) and (v) are counterparts of each other. To run the argument from the previous question, we need a counterpart to the lemma from the previous question. While reading the proof, notice that the proof is the same as that of the Lemma in the previous question, but with the roles of the symbols and cards switched.

Lemma: For any two symbols in a Cobble deck, there is a card that does not contain either of them.

Proof. Let A and B be two symbols, and assume (towards a contradiction) that every card contains A or B (or both). By Property (v), there are four cards, no three of which contain the same symbol. Then two of these cards must contain A and not B, and the other two must contain B and not A. Let K_1 and K_2 be the cards containing A, and let K_3 and K_4 be the cards containing B.

Let C be the unique symbol appearing on K_1 and K_3, and let D be the unique symbol on K_2 and K_4. Let K be the unique card that contains C and D. To complete the proof of the lemma, it is enough to show that K does not contain A or B.

If K contains A, then K and K_1 contain both A and C, so they must be the same card. Also, K and K_2 both contain A and D, so they must also be the same card. But K_1 and K_2 are distinct cards, so this cannot happen and K cannot contain A. A similar argument shows that K cannot contain B, completing the proof of the lemma. □

Great, now everything is in place to run the same argument as in the previous question, but with the roles of the symbols and cards exchanged.

Let A and B be two distinct symbols. Let K be a card not containing A or B, whose existence is guaranteed by the Lemma. Let $K_1, K_2, \ldots, K_n$ be the cards containing A. For each integer i satisfying $1 \leq i \leq n$, let A_i be the unique symbol on K_i and K, and let L_i be the unique card containing A_i and B.

It remains to show that $L_1, L_2, \ldots, L_n$ is the set of cards containing B. First we must show that if $i \neq j$, L_i and L_j are distinct cards. If they are the same card, then A_i and A_j must be the same symbol. This would imply K_i and K_j are the same card, since they both contain A_i and A (which are two distinct symbols since K contains A_i but not A). However, we assumed that if $i \neq j$, K_i and K_j are distinct cards. Therefore, L_i and L_j must be distinct.

Now, to show $L_1, L_2, \ldots, L_n$ is the complete set of cards containing B, suppose L is some card containing B. We want to show that L is the same card as L_i for some i.

Let C be the unique symbol on L and K. Since A is not on K, C and A are distinct. Therefore, there is a unique integer i satisfying $1 \leq i \leq n$ so that K_i contains C and A. Since C is on K_i and K, it must be that $C = A_i$. Then L contains A_i and B, and so $L = L_i$.

We have proved that if one symbol appears on exactly n cards, then so does any other symbol.

To complete the question, we need to show that the number of cards that contain a particular symbol is equal to the number of symbols on a particular card.

To do this, suppose A is a symbol and let K be a card not containing A. Suppose $K = [B_1, B_2, \ldots, B_n]$. We want to show that A is contained in exactly n cards.

For each integer i satisfying $1 \leq i \leq n$, let K_i be the unique card containing A and B_i. Since each of the B_i are distinct, each of the K_i are distinct. Conversely, suppose L is a card containing A. Let B_t be the unique symbol on L and K. Then L contains B_t and A, and so $L = K_t$. Therefore, the list of cards $K_1, K_2, \ldots, K_n$ is precisely the list of cards containing A, and A is contained in exactly n cards.

5. From the previous question, we know that in any Cobble deck, the number of symbols on every card is equal to the number of cards containing any particular symbol. Call this number n. Let's count how many cards are in such a Cobble deck, in terms of n.

   Let $K = [A_1, A_2, \ldots, A_n]$ be a card. By the previous question, each of the A_i are on exactly $n-1$ other cards. This gives $n(n-1)$ other cards in the deck. We claim that these $n(n-1)$ cards, along with K, form all the cards in the deck.

   To see that all these cards are distinct, let L_1 and L_2 be two of the $n(n-1)$ cards. By the way they are constructed, neither are equal to K. Let the unique symbol shared by L_1 and K be A_t, and let the unique symbol shared by L_2 and K be A_s. If $A_t = A_s$, then L_1 and L_2 are distinct by the way they were constructed. If $A_t \neq A_s$, then A_t is on L_1 and A_t is not on L_2, so L_1 and L_2 are distinct.

   To see that these cards form all the cards in the Cobble deck, suppose L is some other card. Then L and K share a unique symbol A_t. Therefore, L is one of the $n-1$ other cards containing A_t, so we have accounted for all the cards in the Cobble deck.

   In terms of n (and adding one for the card K), the total number of cards in the Cobble deck is $n(n-1)+1 = n^2 - n + 1$. So, if there was a Cobble deck with 2026 cards, there would have to be a positive integer n satisfying $n^2 - n + 1 = 2026$. However, there are no integer solutions to this equation (which can be checked by solving the quadratic in n with the quadratic formula). Therefore, there is no Cobble deck with 2026 cards.

Challenge. No solution is given to this, because I do not know of a solution. In fact, no one does. This is an open problem. It is equivalent to the existence of what's called a *finite projective plane of order* 12.

The solution to Question 5 tells us that the number of cards in a Cobble deck must be an integer of the form $n^2 - n + 1$, where n is a positive integer. In Questions 1 and 2 we showed that the smallest n for which there is a Cobble deck is $n = 3$. Cobble decks are known to exist when $n = p^k + 1$ where p is a prime number and k is a positive integer. It is conjectured that these are the only possible values of n for which a Cobble deck exists, but very little has been proven about the nonexistence of Cobble decks.

It is known that Cobble decks do not exist for $n = 7$ and $n = 11$, but that's it! The smallest number for which this question is unsettled is $n = 13$, which corresponds to a Cobble deck with $13^2 - 13 + 1 = 157$ cards.

The game of Dobble (also known as Spot It) is played with what is almost a Cobble deck. The deck consists of cards, each with 8 symbols, and *almost* with each symbol appearing on 8 cards. If a Dobble deck were a Cobble deck, there would be $8^2 - 8 + 1 = 57$ cards in the deck. Inexplicably, there are only 55 cards in a Dobble deck. Go figure!