Problem of the Month
Solution to Problem 5: Stabilising Sequences

February 2026

1. For this sequence we have $a_0^{(1)} = a_1 - a_0 = 8 - 5 = 3$. Since the sequences stabilises at layer 1, we have $a_n^{(1)} = 3$ for all n. Therefore, for all integers $n \geq 0$, $a_{n+1} - a_n = 3$, or equivalently, $a_{n+1} = a_n + 3$. So, each term is three more than the term before it. Since $a_0 = 5$, we have

$$a_1 = 5 + 3$$
$$a_2 = 5 + 2 \cdot 3$$
$$a_3 = 5 + 3 \cdot 3$$
$$a_4 = 5 + 4 \cdot 3$$

and so on. Therefore, $a_{2026} = 5 + 3 \cdot 2026 = 6083$.

2. Let's explicitly compute $a_n^{(k)}$ for the first few values of k. For all integers $n \geq 0$ we have

$$\begin{aligned}
a_n^{(1)} &= a_{n+1} - a_n \\
&= 7(n+1)^4 - 7n^4 \\
&= 7n^4 + 28n^3 + 42n^2 + 28n + 7 - 7n^4 \\
&= 28n^3 + 42n^2 + 28n + 7.
\end{aligned}$$

Then

$$\begin{aligned}
a_n^{(2)} &= a_{n+1}^{(1)} - a_n^{(1)} \\
&= 28(n+1)^3 + 42(n+1)^2 + 28(n+1) + 7 - 28n^3 - 42n^2 - 28n - 7 \\
&= 84n^2 + 84n + 28 + 84n + 48 + 28 + 7 \\
&= 84n^2 + 168n + 111.
\end{aligned}$$

Continuing in this manner,

$$\begin{aligned}
a_n^{(3)} &= a_{n+1}^{(2)} - a_n^{(2)} \\
&= 84(n+1)^2 + 168(n+1) + 111 - 84n^2 - 168n - 111 \\
&= 168n + 84 + 168 \\
&= 168n + 252.
\end{aligned}$$

Finally,

$$\begin{aligned}
a_n^{(4)} &= a_{n+1}^{(3)} - a_n^{(3)} \\
&= 168(n+1) + 252 - 168n - 252 \\
&= 168.
\end{aligned}$$

We have that $a_n^{(4)} = a_{n+1}^{(4)} = 168$ for all integers $n \geq 0$ and $k = 4$.

3. We will take inspiration for this solution from our solution to Question 2. In the solution above, notice that the expression for each sequence $a_n^{(\ell)}$ was always a polynomial, and the degree of the polynomial defining $a_n^{(\ell+1)}$ is one less than the degree of the polynomial defining $a_n^{(\ell)}$. Let's prove this in general.

Suppose for some integer $\ell \geq 0$, $a_n^{(\ell)} = Cn^d + P(n)$, where C is some constant real number other than 0, $d \geq 0$ is an integer, and $P(n)$ is a polynomial in n of degree at most $d-1$.

To understand this notation a little more, let's look at a specific example. If, for example, $a_n^{(\ell)} = 4n^3 + n^2 - n - 1$, we would have $C = 4$, $d = 3$, and $P(n) = n^2 - n - 1$. Notice that $P(n)$ is a polyomial of degree 2 (that is the highest power of n that occurs is 2), and $2 < d$.

Great! So, we assume that $a_n^{(\ell)} = Cn^d + P(n)$. Then

$$a_n^{(\ell+1)} = a_{n+1}^{(\ell)} - a_n^{(\ell)} = C(n+1)^d + P(n+1) - Cn^d - P(n).$$

To proceed, we will need two facts about the behaviour of polynomials. The proofs of these facts will be deferred to the end of this document, but see if you can prove them yourself first!

Fact 1: For all integers $d \geq 1$, $(n+1)^d = n^d + dn^{d-1} + P(n)$ where $P(n)$ is a polynomial of degree at most $d-2$.

Fact 2: If $P(n)$ is a polynomial of degree d, then $P(n+1) - P(n)$ is a polynomial of degree at most $d-1$.

With these facts in hand let's continue. We have

$$a_n^{(\ell+1)} = C(n+1)^d + P(n+1) - Cn^d - P(n)$$
$$= Cn^d + dCn^{d-1} + Q(n) + P(n+1) - Cn^d - P(n)$$
$$= dCn^{d-1} + Q(n) + P(n+1) - P(n)$$

where $Q(n)$ is some polynomial of degree at most $d-2$. Since $P(n)$ has degree at most $d-1$, Fact 2 implies that $P(n+1) - P(n)$ has degree at most $d-2$. Therefore, we can conclude that $R(n) = Q(n) + P(n+1) - P(n)$ is a polynomial of degree at most $d-2$. Putting all of this together, $a_n^{(\ell+1)} = dCn^{d-1} + R(n)$ where $R(n)$ is a polynomial of degree at most $d-2$. We have just proved the following result:

Result 1: If $a_n^{(\ell)} = Cn^d + P(n)$ where $P(n)$ is a polynomial of degree at most $d-1$, then $a_n^{(\ell+1)} = dCn^{d-1} + R(n)$ where $R(n)$ is a polynomial of degree at most $d-2$.

With Result 1 in hand, let's attack the original problem. We are given that $a_n^{(0)} = C_t n^t + P(n)$ for some polynomial $P(n)$ of degree at most $t-1$ (it could be the case that $C_{t-1} = 0$, so the degree may be less than $t-1$). Applying Result 1 repeatedly we get

$$a_n^{(1)} = tC_t n^{t-1} + P_1(n)$$
$$a_n^{(2)} = (t-1)tC_t n^{t-2} + P_2(n)$$
$$a_n^{(3)} = (t-2)(t-1)tC_t n^{t-3} + P_3(n)$$
$$\vdots$$
$$a_n^{(t-1)} = (2)(3)\cdots(t-2)(t-1)tC_t n + P_{t-1}(n)$$

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where $P_r(n)$ is a polynomial of degree at most $t - r - 1$. Since $P_{t-1}(n)$ is a polynomial of degree at most $t - (t - 1) - 1 = 0$, it must just be some real number K. Rewriting our expression for $a_n^{(t-1)}$ a little we have

$$a_n^{(t-1)} = (t!)C_t n + K.$$

We can now explicitly compute $a_n^{(t)} = (t!)C_t(n+1) + K - (t!)C_t n - K = (t!)C_t$ for all n, which is constant! We can conclude that the sequence stabilises at layer t.

4. We will approach this problem in two steps. Step one will be to show that the first four terms determine all the terms in the sequence. Step two will be to use Question 3 to find a general expression for such a sequence that stabilises at layer 3, and has the first four terms given in the statement of the question.

For step one, assume $n \geq 4$. Then

$$
\begin{aligned}
a_{n-4}^{(3)} &= a_{n-3}^{(2)} - a_{n-4}^{(2)} \\
&= a_{n-2}^{(1)} - 2a_{n-3}^{(1)} + a_{n-4}^{(1)} \\
&= a_{n-1} - 3a_{n-2} + 3a_{n-3} - a_{n-4}.
\end{aligned}
$$

Since the sequence given stabilises at layer 3, we have $a_{n-4}^{(3)} = a_{n-3}^{(3)}$. Writing this equality out and rearranging gives

$$
\begin{aligned}
a_{n-1} - 3a_{n-2} + 3a_{n-3} - a_{n-4} &= a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} \\
\Rightarrow \quad a_n &= 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}.
\end{aligned}
$$

This calculation proves that for any sequence that stabilises at layer 3 (not just the one given in the question), each term is determined by the previous four terms. So, for example, we have

$$
\begin{aligned}
a_4 &= 4(43) - 6(13) + 4(5) - 7 = 107 \\
a_5 &= 4(107) - 6(43) + 4(13) - 5 = 217
\end{aligned}
$$

and so on. At this point, you could write a compute program or use a spreadsheet to compute a_{2026}. We're going to do something a little more human!

Since the rest of the sequence is determined by a_0, a_1, a_2, and a_3, if we can find another sequence that stabilises at layer 3 with the same first four values, it must be this sequence! The question is how we find such a sequence. Thankfully, the previous question gives us a way to find it.

From Question 3, if a sequence $b_0, b_1, b_2, \ldots$ is defined by $b_n = An^3 + Bn^2 + Cn + D$ for constants A, B, C, D with $A \neq 0$, then the sequence stabilises at layer 3. Let's see if we can find such constants so that $b_0 = 7$, $b_1 = 5$, $b_2 = 13$, and $b_3 = 43$.

Supposing such constants existed, they would have to satisfy the following system of simultaneous equations:

$$
\begin{aligned}
b_0 &= 7 = A(0)^3 + B(0)^2 + C(0) + D \\
b_1 &= 5 = A(1)^3 + B(1)^2 + C(1) + D \\
b_2 &= 13 = A(2)^3 + B(2)^2 + C(2) + D \\
b_3 &= 43 = A(3)^3 + B(3)^2 + C(3) + D
\end{aligned}
$$

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which after some simplification is equivalent to

$$D = 7$$
$$A + B + C + D = 5$$
$$8A + 4B + 2C + D = 13$$
$$27A + 9B + 3C + D = 43.$$

Subtracting the first equation from each of the other three and rearranging gives

$$C = -2 - A - B$$
$$8A + 4B + 2C = 6$$
$$27A + 9B + 3C = 36.$$

Substituting the first into the other two, rearranging, and simplifying gives

$$3A + B = 5$$
$$4A + B = 7$$

Subtracting the first equation from the second gives $A = 2$. Solving for the remaining variables gives $B = -1$ and $C = -3$.

Putting everything together, we must have that for all $n \geq 0$, $a_n = 2n^3 - n^2 - 3n + 7$. Substituting in $n = 2026$ gives $a_{2026} = 16\,628\,036\,405$.

Proofs of Facts 1 and 2

To keep things neat and tidy, we will declare that the polynomial $P(n) = 0$ has degree -1, and that non-zero constant polynomials have degree 0. You may have come across the convention that $P(n) = 0$ has degree $-\infty$, which would work equally as well. We will stick to the degree being -1.

For Fact 1, we will perform what is called a *proof by induction.*

Notice that the statement we wish to prove is true for $d = 1$ since $(n + 1)^1 = n^1 + 1n^0 + 0$. We now assume that the statement is true for all integers d satisfying $1 \leq d \leq k$, for some positive integer k. The goal now is to show that the assumption implies that the statement is true for $k + 1$.

By our assumption, $(n + 1)^k = n^k + kn^{k-1} + P(n)$ where $P(n)$ has degree at most $k - 2$. Then

$$(n + 1)^{k+1} = (n + 1)(n + 1)^k$$
$$= (n + 1)(n^k + kn^{k-1} + P(n))$$
$$= n^{k+1} + (k + 1)n^k + kn^{k-1} + P(n) + nP(n).$$

The polynomial $nP(n)$ has degree one more than that of $P(n)$. With a bit of thought, see if you can convince yourself that the degree of the sum of two polynomials cannot be larger than both degrees of the original polynomials. Therefore $Q(n) = kn^{k-1} + P(n) + nP(n)$ is a polynomial of degree at most $k - 1$. This proves the desired statement for $k + 1$.

So, we know the statement is true for $d = 1$, and therefore it's true for $d = 2$. Since it's true for $d = 1$ and $d = 2$, we proved that it must be true for $d = 3$. Continuing like this, we can conclude that for all integers $d \geq 1$, $(n + 1)^d = dn^{d-1} + P(n)$ where $P(n)$ has degree at most $d - 2$.

For Fact 2, we will rely on Fact 1. We are given that $P(n)$ has degree d so we can write

$$P(n) = C_d n^d + C_{d-1} n^{d-1} + \cdots + C_1 n + C_0$$

where $C_0, C_1, \ldots, C_d$ are all constants, and $C_d \neq 0$. Then

$$P(n+1) = C_d(n+1)^d + C_{d-1}(n+1)^{d-1} + \cdots + C_1(n+1) + C_0.$$

Applying Fact 1 to expand out $P(n+1)$ allows us to conclude that $P(n+1)$ has degree d. Therefore $P(n+1) - P(n)$ has degree at most d. In order to prove Fact 2, we need to show that the coefficient in front of the n^d term is zero. To that end, let's use Fact 1 to expand out $P(n+1) - P(n)$ a little. We have

$$P(n+1) - P(n) = C_d(n^d + dn^{d-1}) - C_d n^d + Q(n) = dC_d n^{d-1} + Q(n)$$

where $Q(n)$ is a polynomial of degree at most $d-1$. Therefore $P(n+1) - P(n)$ has degree at most $d-1$, completing the proof.