Problem of the Month

Problem 5: Stabilising Sequences

February 2026

Suppose you have an infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers. We can consider the differences between consecutive terms in the sequence to get a new sequence. We can then consider differences between terms in the new sequence, to get yet another sequence, and so on! To keep track of all of this, for all integers $n \geq 0$ and $k \geq 0$ define

$$a_n^{(0)} = a_n \quad \text{and} \quad a_n^{(k)} = a_{n+1}^{(k-1)} - a_n^{(k-1)}.$$

For example, if the original sequence is given by $a_n = n^2$, then

$$\begin{aligned}(a_0, a_1, a_2, a_3, \ldots) = \left(a_0^{(0)}, a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, \ldots\right) &= (0, 1, 4, 9, \ldots) \\ \left(a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots\right) &= (1, 3, 5, 7, \ldots) \\ \left(a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \ldots\right) &= (2, 2, 2, 2, \ldots).\end{aligned}$$

Note that the superscripts (for example the (2) in $a_1^{(2)}$) are *not* exponents, but just notation to keep track of everything.

We say a sequence $a_0, a_1, a_2, \ldots$ *stabilises at layer* k if k is the smallest integer for which $a_n^{(k)} = a_{n+1}^{(k)}$ for all $n \geq 0$. For example, the sequence $a_0, a_1, a_2, \ldots$ defined by $a_n = n^2$ stabilises at layer 2.

1. A sequence that stabilises at layer 1 begins with $a_0 = 5$ and $a_1 = 8$. Compute a_{2026}.
2. The sequence $a_0, a_1, a_2, \ldots$ defined by $a_n = 7n^4$ stabilises at layer k. Find k and compute $a_n^{(k)}$ for every integer $n \geq 0$.
3. A sequence $a_0, a_1, a_2, \ldots$ is defined by $a_n = C_t n^t + C_{t-1} n^{t-1} + C_{t-2} n^{t-2} + \cdots + C_1 n + C_0$ for some integer $t \geq 0$ and some constant real numbers $C_0, C_1, \ldots, C_t$ with $C_t \neq 0$. Show that the sequence stabilises at layer k for some k, and compute $a_n^{(k)}$ for every integer $n \geq 0$.
4. A sequence that stabilises at layer 3 begins with $a_0 = 7$, $a_1 = 5$, $a_2 = 13$, and $a_3 = 43$. Compute a_{2026}.