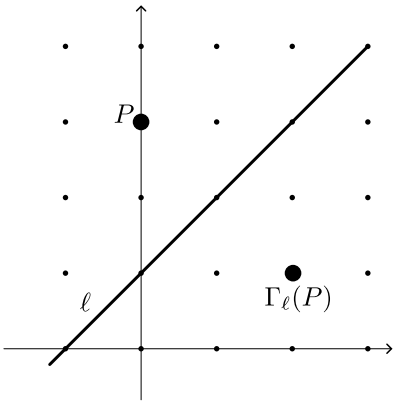
Problem of the Month

Solution to Problem 2: Kaleidoscopes

November 2026

(a) **Solution 1:** We will draw a picture, guess at the answer, and prove that our answer is correct. First, the picture.



The diagram suggests that $\Gamma_\ell(P)$ has coordinates $(2, 1)$.

To see this is the case, let k denote the line passing through $(2, 1)$ and $(0, 3)$. The line k is defined by the equation $y = -x + 3$, and in particular, has slope -1. The slope of ℓ is 1. Since $1 \times -1 = -1$, we know the line passing through $(0, 3)$ and $(2, 1)$ is perpendicular to ℓ.

We now need to show that ℓ bisects the line segment joining $(2, 1)$ and $(0, 3)$. To find the point at which the lines ℓ and k intersect, we solve the system of equations

$$y = x + 1$$
$$y = -x + 3.$$

Solving these equations reveals that the point of intersection is $(1, 2)$. Now, the distance from $(2, 1)$ to $(1, 2)$ is $\sqrt{(2-1)^2 + (1-2)^2} = \sqrt{2}$. Similarly, the distance from $(0, 3)$ to $(1, 2)$ is $\sqrt{2}$. Therefore, ℓ is indeed the perpendicular bisector of the line segment joining $(0, 3)$ and $(2, 1)$. So, by the definition of the reflection Γ_ℓ we have that $\Gamma_\ell(P)$ has coordinates $(2, 1)$.

Solution 2: Unlike Solution 1, we won't guess at the point. We will instead work out algebraically what it must be.

Let k be the line perpendicular to ℓ that passes through $(0, 3)$. Since ℓ has slope 1 and k is perpendicular to ℓ, it must be that k has slope -1. Therefore, the equation of line k is $y - 3 = -1(x - 0)$ or $y = -x + 3$.

The midpoint of P and $\Gamma_\ell(P)$ must lie on ℓ and k. To find the midpoint, we solve the

system of equations

$$y = x + 1$$
$$y = -x + 3.$$

Solving these equations reveals that the midpoint of P and $\Gamma_\ell(P)$ has coordinates $(1, 2)$. Suppose $\Gamma_\ell(P)$ has coordinates (u, v). Then by the midpoint formula we have

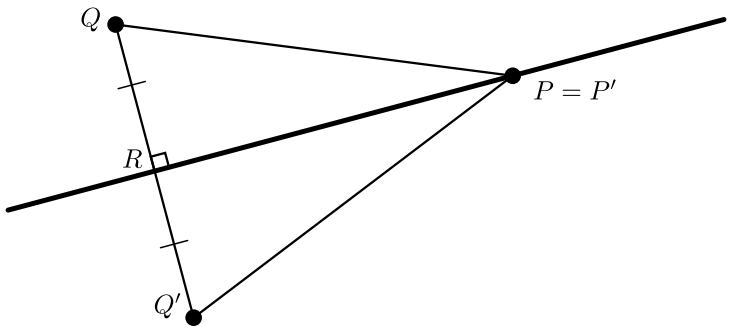
$$\frac{0+u}{2} = 1 \quad \text{and} \quad \frac{3+v}{2} = 2.$$

Solving these equations gives us that $\Gamma_\ell(P)$ has coordinates $(2, 1)$.

(b) First, let's set some notation. For points A and B in the plane, denote the distance between A and B by AB, and the line segment joining A and B by $\overline{AB}$. Denote the reflection of A across a line ℓ by $A' = \Gamma_\ell(A)$. We split up the argument into several cases.

Case 1: P and Q are both on ℓ. In this case, $P = P'$ and $Q = Q'$ so certainly $PQ = P'Q'$.

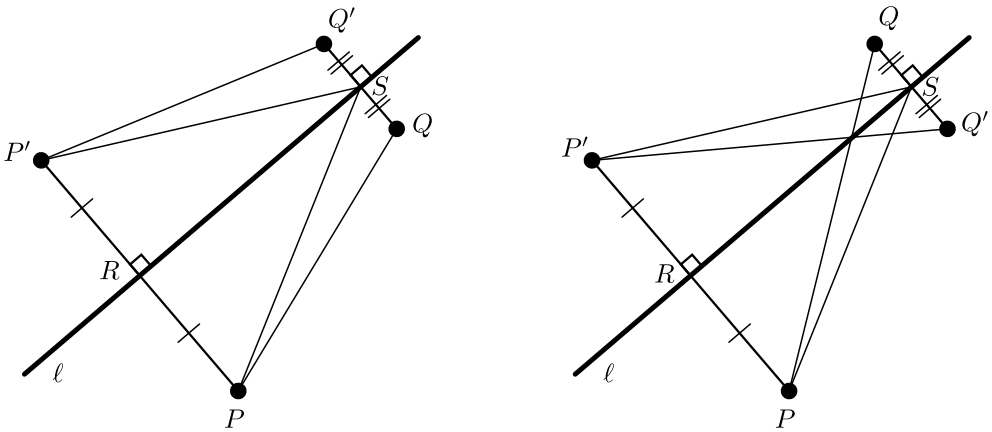
Case 2: P is on ℓ and Q is not on ℓ, as shown in the image.



Note that $P = P'$. Let R be the intersection of ℓ and $\overline{QQ'}$. Then $\triangle PQR$ and $\triangle PQ'R$ are congruent right-angled triangles, and so $PQ = PQ' = P'Q'$.

Case 3: P is not on ℓ and Q is on ℓ. This is the same as Case 2, except with the roles of P and Q switched.

Case 4: P and Q are both not on ℓ. Here we should pay separate attention to when P and Q are on the same side of ℓ, and when they are on opposite sides. Both cases are drawn in the image below. It turns out that the argument is the same for both cases, and applies equally well to both pictures.



Let R and S be the intersections of ℓ with $\overline{PP'}$ and $\overline{QQ'}$ respectively. Similar to the argument in Case 2, $\triangle SRP'$ and $\triangle SRP$ are congruent right-angled triangles. Therefore, $SP' = SP$ and $\angle P'SR = \angle PSR$. Then since $\angle RSQ' = \angle RSQ = 90°$, we have that $\angle P'SQ' = \angle PSQ$. The fact that $Q'S = QS$ implies that $\triangle PSQ$ and $\triangle P'SQ'$ are congruent triangles by side-angle-side congruence. Therefore, $PQ = P'Q'$.

(c) From Part (b) we know that $PQ = P'Q'$, $PR = P'R'$, and $RQ = R'Q'$. Therefore, $\triangle PQR$ and $\triangle P'Q'R'$ are congruent by side-side-side congruence. Thus, $\angle PQR = \angle P'Q'R'$.

(d) The composition $\Gamma_{\ell_2} \circ \Gamma_{\ell_1}$ is the rotation centred at X by 2θ in the counterclockwise direction.

To see this, consider a point P in the plane. If $P = X$, since X lies on both ℓ_1 and ℓ_2, $\Gamma_{\ell_2} \circ \Gamma_{\ell_1}(X) = \Gamma_{\ell_2}(X) = X$, a good start! This is what we want from the rotation centre.

Now suppose P is not X, and let $P' = \Gamma_{\ell_1}(P)$ and $P'' = \Gamma_{\ell_2}(P')$. In order to show that the resulting transformation is indeed a rotation centred at X by 2θ in the counterclockwise direction, we need to show two things:

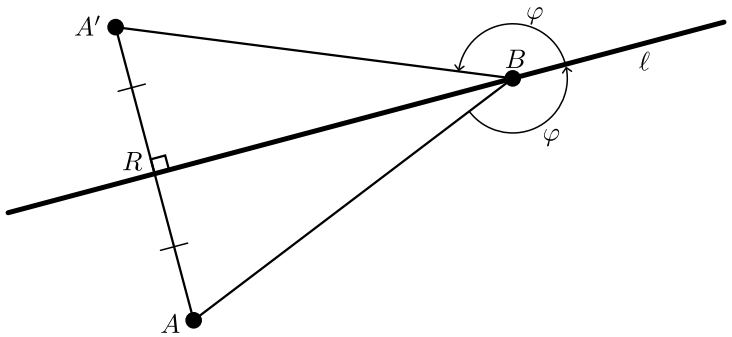
- $XP = XP''$, and
- when measured counterclockwise from $\overline{XP}$, $\angle PXP'' = 2\theta$.

Since X is on ℓ_1 and ℓ_2, we have that $\Gamma_{\ell_1}(X) = X$ and $\Gamma_{\ell_2}(X) = X$. Therefore, by Part (b), $XP = XP' = XP''$.

To show that the second point holds, it will be useful to first prove the following claim:

Claim: Let ℓ be a line in the plane, B a point on ℓ and A a point distinct from B. Let $A' = \Gamma_\ell(A)$. Suppose that, when measured counterclockwise from $\overline{BA}$, the angle between $\overline{BA}$ and ℓ is φ. Then, when measured counterclockwise from ℓ, the angle between ℓ and $\overline{BA'}$ is φ.

To prove this claim, first note that if A is on ℓ, then $\varphi = 0°$. In this case, $A = A'$ and so the angle between ℓ and $\overline{BA'}$ is also $0°$. Suppose then that A is not on ℓ and let R be the point of intersection of $\overline{AA'}$ and ℓ. Similar to the argument in Case 2 of Part (b), $\triangle ABR$ and $\triangle A'BR$ are congruent right-angled triangles as shown in the following image.



Therefore, the angle between $\overline{AB}$ and ℓ is the same as the angle between $\overline{A'B}$ and ℓ, but just measured in opposite directions. This proves the claim.

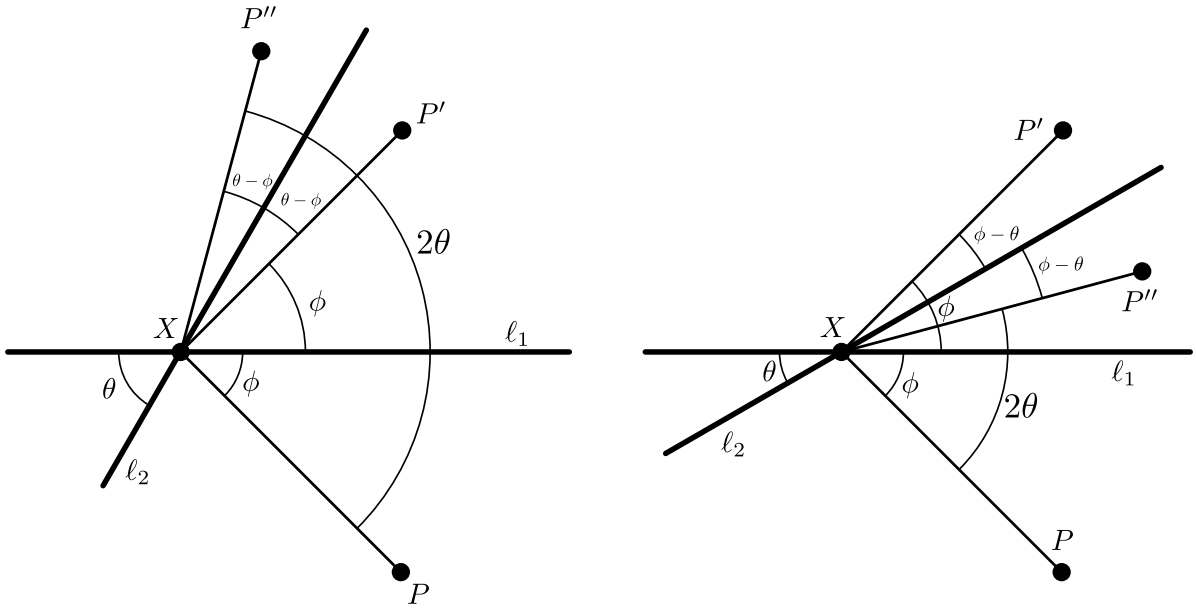
With the claim in hand, suppose the angle between $\overline{XP}$ and ℓ_1, when measured counterclockwise from $\overline{XP}$, is ϕ. Then the angle between $\overline{XP'}$ and ℓ_1, when measured counterclockwise from ℓ_1, is ϕ. Therefore, the angle between ℓ_2 and $\overline{XP'}$, measured counterclock-

wise from ℓ_2, is $\phi - \theta$ (if this number is negative, that just means the angle between $\overline{XP'}$ and ℓ_2 is $|\phi - \theta|$ measured clockwise from ℓ_2).

Applying the claim again, we have that the angle between $\overline{XP''}$ and ℓ_2, when measured counterclockwise from $\overline{XP''}$ is $\phi - \theta$. So, to summarise, we have

- the angle between $\overline{XP}$ and ℓ_1, measured counterclockwise from $\overline{XP}$, is ϕ,
- the angle between ℓ_1 and ℓ_2, measured counterclockwise from ℓ_2, is θ, and
- the angle between ℓ_2 and $\overline{XP''}$, measured counterclockwise from ℓ_2, is $\theta - \phi$.

Here are two images illustrating the situation. The first when $\theta > \phi$, and the second when $\theta < \phi$.



Summing these angles allows us to conclude that the angle between $\overline{XP}$ and $\overline{XP''}$, measured counterclockwise from $\overline{XP}$, is $\phi + \theta + \theta - \phi = 2\theta$ as desired.

(e) In the solution to Part (e) we will use the following notation. For a non-negative integer s and any transformation T of the plane, T^s will denote the composition of T with itself s times. For example, $\Gamma_{\ell_1}^2 = \Gamma_{\ell_1} \circ \Gamma_{\ell_1}$, and $(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^3 = \Gamma_{\ell_2} \circ \Gamma_{\ell_1} \circ \Gamma_{\ell_2} \circ \Gamma_{\ell_1} \circ \Gamma_{\ell_2} \circ \Gamma_{\ell_1}$. If $s = 0$, we will take it to mean that no transformation is applied. For example, $\Gamma_{\ell_1}^0(A) = A$ for all points A in the plane.

(i) At first glance, it may appear that we get infinitely many points, since we can keep applying the reflections forever. However, there are two important observations we can make that will help us deduce that there are only finitely many points.

Observation 1. For any point A in the plane and any reflection Γ_ℓ, we have that $\Gamma_\ell^2(A) = A$. Said another way, if you reflect and then reflect again, you come back to where you started!

Observation 2. As a consequence of Part (d), the composition $\Gamma_{\ell_2} \circ \Gamma_{\ell_1}$ is the rotation counterclockwise about the point X by $2\left(\frac{180^\circ}{n}\right) = \frac{360^\circ}{n}$. Therefore, for every point A in the plane,

$$(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^n(A) = A.$$

Now we can turn our attention to the question at hand. An arbitrary point in the orbit of P (other than P itself) is of the form,

$$\Gamma_k \circ \Gamma_{k-1} \circ \cdots \circ \Gamma_1(P),$$

for some positive integer k, where each Γ_i is either Γ_{ℓ_1} or Γ_{ℓ_2}. Since P is on ℓ_2, if Γ_1 is Γ_{ℓ_2}, then we have

$$\Gamma_k \circ \Gamma_{k-1} \circ \cdots \circ \Gamma_2 \circ \Gamma_1(P) = \Gamma_k \circ \Gamma_{k-1} \circ \cdots \circ \Gamma_2(P).$$

Therefore, since we are trying to find all distinct points in the orbit of P, we need only to look at points of the form $\Gamma_k \circ \Gamma_{k-1} \circ \cdots \circ \Gamma_2 \circ \Gamma_{\ell_1}(P)$.

By Observation 1 above, we can rule out any appearance of two consecutive appearances of the same reflection. Therefore, each point (other than P itself) is a composition of reflections alternating between Γ_{ℓ_1} and Γ_{ℓ_2} applied to P, where the first reflection applied is Γ_{ℓ_1}. So, any point in the orbit of P is of the form

$$(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^m(P) \quad \text{or} \quad (\Gamma_{\ell_1} \circ \Gamma_{\ell_2})^m \circ \Gamma_{\ell_1}(P)$$

for some non-negative integer m.

Let's first focus on the points $(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^m(P)$. If $m \geq n$, then we can write $m = qn + r$ for some positive integer q and some integer r satisfying $0 \leq r < n$ (this observation is commonly called the *division algorithm* and just says that n divides m q times with a remainder of r). Then, by applying Observation 2 repeatedly we get

$$\begin{aligned}(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^m(P) &= (\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^r \circ \underbrace{(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^n \circ (\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^n \cdots (\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^n}_{q\text{-times}}(P) \\ &= (\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^r(P).\end{aligned}$$

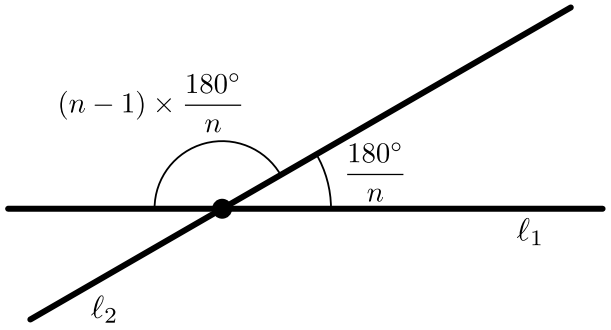
Therefore, any point of the form $(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^m(P)$ is one of the n points

$$\begin{aligned}&P,\\ &(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})(P),\\ &(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^2(P),\\ &\vdots\\ &(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^{n-2}(P), \text{ or}\\ &(\Gamma_{\ell_2} \circ \Gamma_{\ell_1})^{n-1}(P).\end{aligned}$$

Note that these n points are precisely the points where P is ends up after a rotation about X counterclockwise by $0°, \frac{360°}{n}, 2 \times \frac{360°}{n}, \ldots, (n-2) \times \frac{360°}{n}$, and $(n-1) \times \frac{360°}{n}$, respectively. Therefore, this list of n points is a list of n *distinct* points. The goal is to now show that these are all the points in the orbit of P.

The only other possible points are of the form $(\Gamma_{\ell_1} \circ \Gamma_{\ell_2})^m \circ \Gamma_{\ell_1}(P)$ for some non-negative integer m.

Let's focus on the transformation $\Gamma_{\ell_1} \circ \Gamma_{\ell_2}$. Since, when measured counterclockwise from ℓ_1, the angle between the two lines is $\frac{180°}{n}$, when measured counterclockwise from ℓ_2, the angle between the lines is $180° - \frac{180°}{n} = (n-1) \times \frac{180°}{n}$, as shown in the next image.



Therefore, by Part (d), $\Gamma_{\ell_1}\circ\Gamma_{\ell_2}$ is the rotation counterclockwise about X by an angle of

$$2(n-1)\times\frac{180^\circ}{n}=360^\circ-\frac{360^\circ}{n}.$$

This is the same as the rotation *clockwise* about X by an angle of $\frac{360^\circ}{n}$, which is equal to the transformation $(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{n-1}$. So,

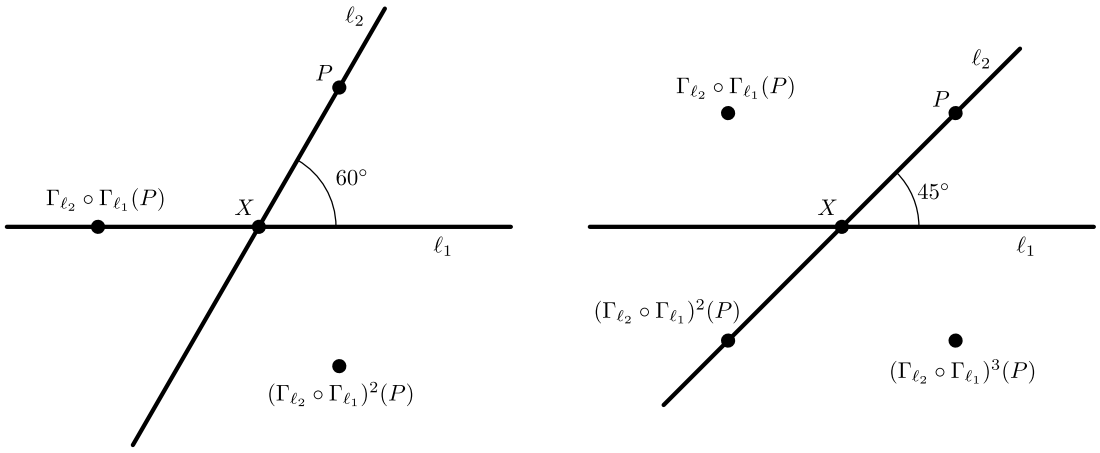
$$\begin{aligned}(\Gamma_{\ell_1}\circ\Gamma_{\ell_2})^m\circ\Gamma_{\ell_1}(P) &= \underbrace{(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{n-1}\circ\cdots\circ(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{n-1}}_{m\text{-times}}\circ\Gamma_{\ell_1}(P)\\ &= (\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{m(n-1)}\circ\Gamma_{\ell_1}(P)\\ &= (\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{m(n-1)-1}\circ\Gamma_{\ell_2}\circ\Gamma_{\ell_1}\circ\Gamma_{\ell_1}(P)\\ &= (\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{m(n-1)-1}\circ\Gamma_{\ell_2}(P)\\ &= (\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{m(n-1)-1}(P).\end{aligned}$$

However, we have already shown that a point of the form $(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^l(P)$ is one of the n points listed above.

Therefore, we have that the orbit of P consists of the n points

$$P,(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})(P),(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^2(P),\ldots,(\Gamma_{\ell_2}\circ\Gamma_{\ell_1})^{n-1}(P).$$

Here is a picture of the points in the orbit of P when $n=3$ and $n=4$.



(ii) Let R denote the rotation centred at X by an angle of $\frac{360^\circ}{n}$ counterclockwise. From the previous part we know that the orbit of P consists of the n points

$$P,\ R(P),\ R^2(P),\ \ldots,\ \text{and } R^{n-1}(P).$$

Call these points $P_0, P_1, P_2, \ldots, P_{n-1}$ respectively. Observe that $R(P_{n-1}) = P_0$ and if $0 \leq i \leq n-2$, then $R(P_i) = P_{i+1}$. Note that R is the composition of two reflections. Since reflections preserve distance between points (by Part (b)), we have that

$$P_{n-2}P_{n-1} = P_{n-1}P_0 = P_0P_1 \quad \text{and} \quad P_iP_{i+1} = P_{i+1}P_{i+2}$$

for all i such that $0 \leq i \leq n-3$. This shows that the n-gon $P_0P_1P_2 \cdots P_{n-1}$ has equal side lengths. As a consequence of Part (c), we know that R preserves angles. Therefore,

$$\angle P_{n-3}P_{n-2}P_{n-1} = \angle P_{n-2}P_{n-1}P_{n-0} = \angle P_{n-1}P_0P_1 = \angle P_0P_1P_2$$

and

$$\angle P_iP_{i+1}P_{i+2} = \angle P_{i+1}P_{i+2}P_{i+3}$$

for all i satisfying $0 \leq i \leq n-4$. We can finally conclude that polygon $P_0P_1P_2 \cdots P_{n-1}$ is a regular n-gon.