Problem of the Month

Problem 2: Kaleidoscopes

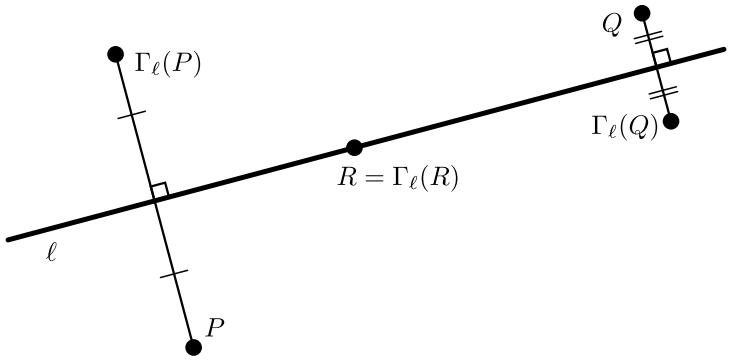
November 2026

A kaleidoscope is a toy made from mirrors that intersect at particular angles, and the resulting reflections create mesmorising patterns. In this month's problem, you will investigate how reflections interact with each other.

Given a line ℓ in the plane, the *reflection across* ℓ, which we will denote by Γ_ℓ, is a transformation of the plane that sends every point in the plane to another point as follows:

If P is on ℓ, then $\Gamma_\ell(P) = P$ (the point isn't affected by the reflection). If P is not on ℓ, then $\Gamma_\ell(P)$ is the unique point in the plane so that ℓ is the perpendicular bisector of the line segment joining P and $\Gamma_\ell(P)$ (you may take it for granted that there is indeed such a unique point).

The next image illustrates a line ℓ with points P, Q, and R and their reflections across ℓ.



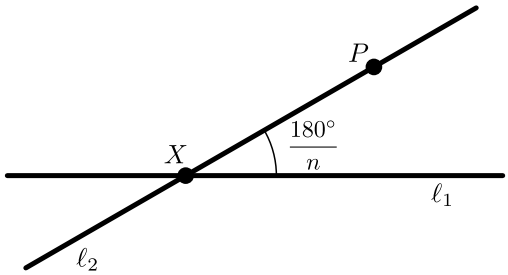
(a) If ℓ is the line defined by the equation $y = x + 1$, and P has coordinates $(0, 3)$, what are the coordinates of $\Gamma_\ell(P)$?

(b) Let P and Q be points in the plane and let ℓ be a line. Show that the distance between P and Q is equal to the distance between $\Gamma_\ell(P)$ and $\Gamma_\ell(Q)$.

(c) Let P, Q and R be points in the plane. Let ℓ be a line and let $P' = \Gamma_\ell(P)$, $Q' = \Gamma_\ell(Q)$ and $R' = \Gamma_\ell(R)$. Show that $\angle PQR = \angle P'Q'R'$.

(d) Let ℓ_1 and ℓ_2 be two lines in the plane that intersect at a point X. Suppose that when measured counterclockwise from ℓ_1, the angle formed by the two lines is θ. Describe geometrically the transformation obtained by the composition $\Gamma_{\ell_2} \circ \Gamma_{\ell_1}$ (that is, the transformation resulting from the reflection across ℓ_1 followed by the reflection across ℓ_2).

(e) Let ℓ_1 and ℓ_2 be two lines in the plane that intersect at a point X. Suppose that when measured counterclockwise from ℓ_1, the angle formed by the two lines is $\frac{180^\circ}{n}$ for some integer $n \geq 3$. As shown below, let P be a point on ℓ_2 other than X.



By repeatedly applying the reflections Γ_{ℓ_1} and Γ_{ℓ_2} to P, as many times as you like and in any order, you will obtain a set of points. For example, $P, \Gamma_{\ell_2}(P), \Gamma_{\ell_1} \circ \Gamma_{\ell_2}(P)$, and $\Gamma_{\ell_1} \circ \Gamma_{\ell_2} \circ \Gamma_{\ell_1}(P)$ are four such points. Some of the points obtained this way will be the same. Call the set of all points obtained this way the *orbit of* P.

(i) Including the point P, how many distinct points are in the orbit of P?

(ii) Let k be the answer to Part (i). Show that the k points in the orbit of P form the vertices of a regular k-gon. [*Recall that a regular k-gon is a polygon with k equal side lengths and equal internal angles*].