



Problem of the Month

Problem 1: Pairing socks

October 2026

Suppose n is a positive integer and that you have n pairs of socks. Within each pair of socks, the two socks are the same colour. Every pair has a unique colour. After doing laundry, all of the $2n$ socks are in a laundry basket. You begin to remove socks one at a time (always choosing randomly and never replacing the socks) until you find a pair. That is, you remove socks until you remove a sock that matches a sock that has already been removed.

For positive integers n and k with $k < 2n$, we define $P(n, k)$ to be the probability that the first k socks removed are all different colours and there is a pair among the first $k + 1$ socks that are removed. That is, $P(n, k)$ is the probability that the first pair is found upon removing the $(k + 1)^{\text{st}}$ sock.

- (a) Compute $P(4, k)$ for each k from 1 through 7. Some of these probabilities should equal 0.
- (b) Find a general formula for $P(n, k)$ when $k \leq n$. It might be useful later if you can find a formula that only uses addition, subtraction, multiplication, division, exponentiation, as well as factorials and binomial coefficients. Notice that the question does not ask about $P(n, k)$ for $k > n$. You might want to think about what happens in this case.
- (c) For a positive integer i , define T_i to be the sum of the first i positive integers. For example, $T_1 = 1$, $T_2 = 1 + 2 = 3$, and $T_3 = 1 + 2 + 3 = 6$. The numbers T_1, T_2, T_3, T_4 , and so on, are often called the *triangular numbers*. You may already know that $T_i = \frac{i(i+1)}{2}$ for every positive integer i . If not, think about why!
 - (i) Suppose $n = T_i$ for some $i > 1$. Show that the largest value of $P(n, k)$ is achieved when $k = i$ and when $k = i + 1$.
 - (ii) Suppose n is a positive integer that is not a triangular number. This means n is strictly between T_i and T_{i+1} for some i . Show that the largest value of $P(n, k)$ is achieved when

$$k = \left\lfloor \frac{1 + \sqrt{8n + 1}}{2} \right\rfloor.$$

- (d) For a positive integer n , we call k a *peak* for n if $P(n, k) \geq P(n, \ell)$ for all integers $1 \leq \ell \leq n$. Part (c) suggests that there are two peaks for n when $n > 1$ is a triangular number and that there is a unique peak when n is not a triangular number. Find a positive integer k for which there are exactly 2026 positive integers n with the property that k is a peak for n .