



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2026 Hypatia Contest

Wednesday, April 1, 2026
(in North America and South America)

Thursday, April 2, 2026
(outside of North America and South America)

Solutions

1. (a) Since A , B and C are collinear, segments AB and AC (and BC) have the same slope. The slope of segment AB is $\frac{5-2}{2-1} = 3$. The slope of segment AC is $\frac{c-2}{3-1} = \frac{c-2}{2}$. Therefore, we have that $3 = \frac{c-2}{2}$. Solving for c gives us that $c = 8$.
- (b) Since segment DF has slope $\frac{7-0}{0-14} = -\frac{1}{2}$ and y -intercept 7, the equation of the line through points D and F is $y = -\frac{1}{2}x + 7$. The coordinates (x, y) of E must be positive integers and E must lie on the line $y = -\frac{1}{2}x + 7$. When $x \geq 14$ we have that $y \leq 0$, and when $x < 14$ we have that $y > 0$. Therefore x and y are both positive exactly when $0 < x < 14$. Also, y is an integer exactly when x is even. Therefore the number of possible points E is the number of integers $0 < x < 14$ such that x is even. There are 6 such integers which give the following 6 possibilities for E : $(2, 6)$, $(4, 5)$, $(6, 4)$, $(8, 3)$, $(10, 2)$, and $(12, 1)$.
- (c) The slope of segment PQ is $\frac{n-4-12}{6-15} = \frac{n-16}{-9}$. The slope of segment PR is $\frac{n-12}{18-15} = \frac{n-12}{3}$. Therefore, points P , Q and R are collinear exactly when $\frac{n-16}{-9} = \frac{n-12}{3}$. This is equivalent to $3n - 48 = -9n + 108$, which is equivalent to $12n = 156$. Therefore, the value of n that makes P , Q and R collinear is $n = \frac{156}{12} = 13$.
2. (a) Since $0 < A < B < C < 10$, the largest possible value of A is 7, and when $A = 7$, we must have $B = 8$ and $C = 9$. Therefore the largest possible peak number is 78 987. Similarly, the smallest possible peak number is 12 321. Therefore the positive difference is $78\,987 - 12\,321 = 66\,666$.
- (b) Since $36\,245 < ABCBA < 45\,932$, we have $A = 4$ or $A = 3$. If $A = 4$, then the first inequality is true for all possible values of B and C , so we only need to consider $4BCB4 < 45\,932$. We must then have $B = 5$, and then C can be any of 6, 7, or 8, so there are 3 peak numbers when $A = 4$. (They are 45 654, 45 754, 45 854.) If $A = 3$, then the second inequality is true for all possible values of B and C , so we only need to consider $36\,245 < 3BCB3$. Then B can be 6, in which case $7 \leq C \leq 9$, or $B = 7$, in which case $8 \leq C \leq 9$, or $B = 8$, in which case we have that $C = 9$. Altogether, there are 6 peak numbers when $A = 3$. (They are 36 763, 36 863, 36 963, 37 873, 37 973, 38 983.) Putting both cases together, there are a total of $3 + 6 = 9$ peak numbers satisfying the desired inequalities.
- (c) A number is a multiple of 15 exactly when it is a multiple of both 5 and 3, and a number is a multiple of 5 exactly when it has units digit 0 or 5. Since $0 < A < 10$ and $ABCBA$ is a multiple of 15, we must have $A = 5$. Furthermore, $5BCB5$ is a multiple of 15 exactly when $5BCB5$ is a multiple of 3. A positive integer is a multiple of 3 exactly when the sum of its digits is a multiple of 3. That is, $5BCB5$ is a multiple of 3 exactly when $10 + 2B + C$ is a multiple of 3. Since $A = 5$ and $A < B < C$, the possible values of B are 6, 7 and 8.

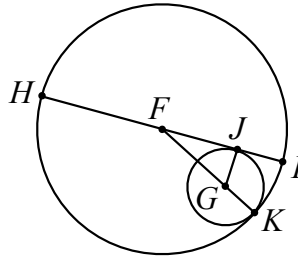
When $B = 6$, $10 + 12 + C$ is a multiple of 3 precisely when $C = 8$ (among the possible values 7, 8, and 9 of C).

When $B = 7$, $10 + 14 + C$ is a multiple of 3 precisely when $C = 9$.

Finally, when $B = 8$, the only possible value of C is 9, and $10 + 16 + 9$ is not a multiple of 3.

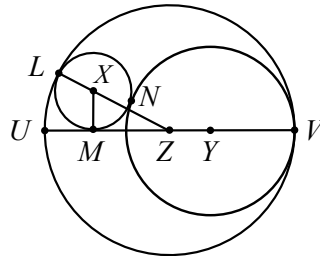
Therefore, the peak numbers that are a multiple of 15 are 56 865 and 57 975.

3. (a) Since DC is a diameter of the circle with radius 36, we have that $DC = 36(2) = 72$. Similarly, $CE = 10(2) = 20$.
By Fact 1, we have that $DE = CD - CE = 72 - 20 = 52$.
- (b) Extend segment FG to K , noting that these three points lie on the same line by Fact 1. Also draw in segment JG . These constructions are shown in the diagram below.

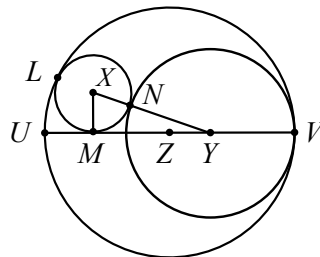


We have that $FG = FK - GK = 49 - 12 = 37$. By Fact 2, we have that $\angle GJF = 90^\circ$. By the Pythagorean Theorem $FG^2 = JG^2 + FJ^2$. That is, $37^2 = 12^2 + FJ^2$. Therefore $FJ^2 = 1225$, and so $FJ = 35$ (since $FJ > 0$).

- (c) We will continue to use Facts 1 and 2, but we will no longer cite them each time. First draw in line segment ZL (noting it passes through X), and then draw in segment XM . These constructions are shown below.



We have that $XZ = LZ - LX = 18 - 5 = 13$. Since $XM = 5$, by the Pythagorean Theorem we deduce that $MZ = 12$. With this result, erase line segment ZL from the above diagram and draw in line segment XY , giving us the diagram below.



Note that $18 = ZV = ZY + YV = ZY + r$ and therefore $ZY = 18 - r$. Therefore $MY = MZ + ZY = 12 + (18 - r) = 30 - r$.

Note that $XY = XN + NY = 5 + r$. Now apply the Pythagorean Theorem to $\triangle XMY$.

We have the following equivalent equations:

$$\begin{aligned} XM^2 + MY^2 &= XY^2 \\ 5^2 + (30 - r)^2 &= (5 + r)^2 \\ 25 + 900 - 60r + r^2 &= 25 + 10r + r^2 \\ 70r &= 900 \\ r &= \frac{90}{7} \end{aligned}$$

which is our final answer.

4. (a) From (*) we have that $y^2 - z^2 = (m + 2)^2 - (m + 1)^2$ and $x^2 - y^2 = (m + 1)^2 - m^2$. We then have that

$$\begin{aligned} k &= (y^2 - z^2) - (x^2 - y^2) \\ &= ((m + 2)^2 - (m + 1)^2) - ((m + 1)^2 - m^2) \\ &= (m^2 + 4m + 4 - m^2 - 2m - 1) - (m^2 + 2m + 1 - m^2) \\ &= (2m + 3) - (2m + 1) \\ &= 2 \end{aligned}$$

We showed that $y^2 - z^2 = 2m + 3$ and $x^2 - y^2 = 2m + 1$ and since m is a positive integer it follows that $z < y < x$, which we will use in part (b).

- (b) We begin with a useful fact: the sum of the first n odd positive integers is equal to n^2 . That is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Note: *Contestants were not required to justify this, so we save a justification until the end.* We are given that $y^2 - z^2$ is the sum of p consecutive odd integers, the largest of which is $2y - 1$. Therefore,

$$\begin{aligned} y^2 - z^2 &= \overbrace{(2y - 1) + (2y - 3) + (2y - 5) + \cdots + (2y - (2p - 1))}^{p \text{ terms}} \\ &= 2yp - (1 + 3 + 5 + \cdots + (2p - 1)) \\ &= 2yp - p^2 \end{aligned}$$

where we used the useful fact in the final step. Therefore $z^2 = y^2 - 2yp + p^2 = (y - p)^2$, so $z = y - p$ or $-z = y - p$. We will show in fact that $y - z = p$ (and thus $z = y - p$). To see this, note that $y^2 = 1 + 3 + 5 + \cdots + (2y - 1)$ expresses y^2 as a sum of y odd integers, and $z^2 = 1 + 3 + 5 + \cdots + (2z - 1)$ expresses z^2 as a sum of z odd integers, and therefore

$$\begin{aligned} y^2 - z^2 &= 1 + 3 + 5 + \cdots + (2y - 1) - (1 + 3 + 5 + \cdots + (2z - 1)) \\ &= \text{sum of } y - z \text{ consecutive odd integers, the largest of which is } 2y - 1 \\ &\quad (\text{since } y > z \text{ as shown in part (a)}) \end{aligned}$$

But $y^2 - z^2$ is also the sum of p consecutive odd integers, the largest of which is $2y - 1$, and so $y - z = p$. Similar calculations yield $x^2 - y^2 = 2xq - q^2$ and $x - y = q$ or $x = y + q$.

We now finish this question with a series of equivalent equations.

$$\begin{aligned}y^2 - z^2 &= x^2 - y^2 + 2 && \text{from part (a)} \\2yp - p^2 &= 2xq - q^2 + 2 \\2yp - p^2 &= 2(y + q)q - q^2 + 2 \\2yp - p^2 &= 2yq + 2q^2 - q^2 + 2 \\2y(p - q) &= p^2 + q^2 + 2 \\y &= \frac{p^2 + q^2 + 2}{2(p - q)}\end{aligned}$$

For the final step, note that $p - q \neq 0$, since if $p = q$, the second to last equation would give $0 = p^2 + p^2 + 2$. This is equivalent to $p^2 = -1$, which is not possible for an integer p . We have obtained an expression for y in terms of p and q .

- (c) We just established that $y = \frac{p^2 + q^2 + 2}{2(p - q)}$. We begin by finding some values of p and q for which y is a positive integer. If $p - q$ is odd, then one of p, q is even and the other is odd, and then $p^2 + q^2 + 2$ is odd. Since the denominator $2(p - q)$ is even, it follows that y is not an integer when $p - q$ is odd. Therefore $p - q$ is even. Suppose $p - q = 2$. Then $p = q + 2$ and

$$y = \frac{p^2 + q^2 + 2}{2(p - q)} = \frac{(q + 2)^2 + q^2 + 2}{2((q + 2) - q)} = \frac{q^2 + 4q + 4 + q^2 + 2}{4} = \frac{q^2 + 2q + 3}{2}$$

If q is even, then the numerator is odd, and therefore y is not an integer. Thus we must consider odd values of q . From (b) we know that $z = y - p = y - (q + 2)$ and $x = y + q$. Furthermore, we have that $m^2 + x^2 = (m + 1)^2 + y^2$. Solving for m (the m^2 terms subtract out) yields that $m = \frac{1}{2}(x^2 - y^2 - 1)$. In the table below, for some odd values of q , we determine x, y, z , and m :

q	x	y	z	m
1	4	3	0	3
3	12	9	4	31
5	24	19	12	107

When $q = 1$, we have that $3^2 + 4^2 = 4^2 + 3^2 = 5^2 + 0^2$, but we reject $S = 25$ since we require $z > 0$.

When $q = 3$, we have that $31^2 + 12^2 = 32^2 + 9^2 = 33^2 + 4^2 = 1105$.

When $q = 5$, we have that $107^2 + 24^2 = 108^2 + 19^2 = 109^2 + 12^2 = 12\,025$.

Therefore 1105 and 12025 are two Sorensen numbers.

Editorial Comment: It turns out that 1105 and 12025 are the smallest two Sorensen numbers and every odd $q \geq 3$ results in a corresponding Sorensen number.

In part (b), we stated and used a useful fact that we will now restate and prove for completeness.

Useful Fact: The sum of the first n odd positive integers is equal to n^2 .

Using the more familiar fact that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

we have that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n-1) + n &= (1+1) + (3+1) + (5+1) + \cdots + (2n-1+1) \\ &= 2 + 4 + 6 + \cdots + 2n \\ &= 2[1 + 2 + 3 + \cdots + n] \\ &= 2 \frac{n(n+1)}{2} \\ &= n^2 + n \end{aligned}$$

Subtracting n from both sides results in the desired identity, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.