



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2026 Galois Contest

Wednesday, April 1, 2026
(in North America and South America)

Thursday, April 2, 2026
(outside of North America and South America)

Solutions

1. (a) The three smallest prime numbers are 2, 3 and 5. Their product is $2 \times 3 \times 5 = 30$.
- (b) Each prime number is a positive integer, and so $\frac{c-3}{4}$ cannot be a prime number unless $c > 3$. The value of $\frac{c-3}{4}$ is an integer exactly when $c-3$ is divisible by 4. The values of c for which $4 \leq c \leq 20$ and $c-3$ is divisible by 4 are 7, 11, 15, and 19. When c is equal to 7 and 19, the values of $\frac{c-3}{4}$ are 1 and 4, respectively. Each of these is not a prime number.
- When $c = 11$, the value of expression is $\frac{11-3}{4} = 2$, which is a prime number.
- When $c = 15$, the value of expression is $\frac{15-3}{4} = 3$, which is a prime number.
- Thus, the two possible values of the integer c are 11 and 15.
- (c) Factoring the given expression, we get $21d - 77 = 7(3d - 11)$. For all integers d , the value of $3d - 11$ is equal to an integer, and so $7(3d - 11)$ is the product of two integers, 7 and $3d - 11$. A prime number is positive and cannot be written as a product of two integers both greater than 1, so $3d - 11 = 1$. If $3d - 11 = 1$, then $3d = 12$, and so $d = 4$. The only integer d with $1 \leq d \leq 10$ for which $21d - 77$ is a prime number is $d = 4$, and the prime number is $21(4) - 77 = 7$.
- We can confirm that if $d < 4$, $3d - 11$ is negative and so $7(3d - 11)$ is not prime. Further, if $d > 4$, $3d - 11 \geq 3(5) - 11 = 4$ and so $7(3d - 11)$ is the product of two integers, both greater than 1, and thus is also not prime.
2. (a) Since $B(2, 1)$ and $C(6, 1)$ lie along the same horizontal line, then $BC = 6 - 2 = 4$, which is the positive difference between their x -coordinates. Suppose G is the point that lies on BC vertically below $A(3, 4)$. Then G has the same x -coordinate as $A(3, 4)$ and the same y -coordinate as $B(2, 1)$ and $C(6, 1)$, and thus has coordinates $(3, 1)$. Since $A(3, 4)$ and $G(3, 1)$ lie along the same vertical line, then $AG = 4 - 1 = 3$, the positive difference between their y -coordinates. The area of $\triangle ABC$ is $\frac{1}{2} \times (BC) \times (AG) = \frac{1}{2} \times 4 \times 3 = 6$.
- (b) The reflection of the point (x, y) in the y -axis is the point $(-x, y)$. Thus, the reflection of $B(2, 1)$ in the y -axis is $D(-2, 1)$. Using $G(3, 1)$ from part (a), we get $DC = 6 - (-2) = 8$, $AG = 3$, and so the area of $\triangle ADC$ is $\frac{1}{2} \times (DC) \times (AG) = \frac{1}{2} \times 8 \times 3 = 12$.
- (c) Point $A(3, 4)$ lies $4 - (-2) = 6$ units vertically above the line $y = -2$. So, the reflection of A in the line $y = -2$ lies 6 units vertically below the line $y = -2$, and thus has coordinates $E(3, -2 - 6)$ or $E(3, -8)$. We may once again use $G(3, 1)$ from part (a) since G lies on BC and lies on the vertical line through $E(3, -8)$. Doing so, we get $EG = 1 - (-8) = 9$ and $BC = 4$. Thus, the area of $\triangle EBC$ is $\frac{1}{2} \times (BC) \times (EG) = \frac{1}{2} \times 4 \times 9 = 18$.

(d) Using $G(3, 1)$ from part (a), $\triangle FBC$ has base $BC = 4$, height FG , and area 12.

Since the area of $\triangle FBC$ is $\frac{1}{2} \times (BC) \times (FG) = 12$, then $2 \times (FG) = 12$, and so the triangle has height $FG = 6$.

With $FG = 6$ and F vertically above $G(3, 1)$, we determine that F has coordinates $(3, 1 + 6) = (3, 7)$. With $FG = 6$ and F vertically below $G(3, 1)$, we determine that F has coordinates $(3, 1 - 6) = (3, -5)$.

Since F is the image of the point $A(3, 4)$ after it is reflected in the horizontal line $y = k$, then the distance from the line to F is equal to the distance from the line to A .

This tells us that k is equal to the average of the y -coordinates of F and A .

The average of the y -coordinates of $F(3, 7)$ and $A(3, 4)$ is $\frac{7 + 4}{2} = \frac{11}{2}$.

The average of the y -coordinates of $F(3, -5)$ and $A(3, 4)$ is $\frac{-5 + 4}{2} = -\frac{1}{2}$.

The two values of k for which the area of $\triangle FBC$ is 12 are $k = \frac{11}{2}$ and $k = -\frac{1}{2}$.

3. (a) Since we are given that the cycle length for the units digits of powers of 3 is equal to 4, and $43 = 4 \times 10 + 3$, the units digit of 3^{43} is equal to the units digit of 3^3 , which is 7.
- (b) Each integer multiple of 10 has units digit 0, and so we begin by determining the repeating sequence of units digits for powers of 4 and 8.

$$4^1 = 4, 4^2 = 16, 4^3 = 64, \dots \quad 8^1 = 8, 8^2 = 64, 8^3 = 512, 8^4 = 4096, 8^5 = 32768, \dots$$

For powers of 4, the consecutive units digits that repeat are 4, 6, with cycle length 2.

For powers of 8, the consecutive units digits that repeat are 8, 4, 2, 6, with cycle length 4.

We can determine the units digit of $4^j + 8^j$ by adding the corresponding units digits of the individual powers of 4 and 8, and then taking the units digit of that sum.

Thus, the units digits of $4^j + 8^j$ are the units digits of $4 + 8 = 12$, $6 + 4 = 10$, $4 + 2 = 6$, $6 + 6 = 12$, which are 2, 0, 6, 2, and this sequence continues to repeat with cycle length 4. So, $4^j + 8^j$ is a multiple of 10 (has units digit 0) once every 4 consecutive values of j .

Since $2026 = 4 \times 506 + 2$, and 0 is the second digit in the repeating sequence 2, 0, 6, 2, then the number of integers j with $1 \leq j \leq 2026$ for which $4^j + 8^j$ is a multiple of 10 is $506 + 1 = 507$.

- (c) For 2^k , the consecutive units digits that repeat are 2, 4, 8, 6 with cycle length 4.
 For 3^k , the consecutive units digits that repeat are 3, 9, 7, 1, with cycle length 4.
 Similar to how we worked with $4^j + 8^j$ in part (b), the consecutive units digits of $2^k + 3^k$ that repeat are 5, 3, 5, 7, with cycle length 4.

For 8^k , the consecutive units digits that repeat are 8, 4, 2, 6, with cycle length 4.

When k is even, that is when $k = 2m$ for positive integers m , $2026k = 2026 \times 2m = 4052m$. Since $4052 = 4 \times 1013$, then $4052m$ is a multiple of 4, and so $2026k$ is a multiple of 4 for all even integers k .

Thus for all even integers k , the units digit of 8^{2026k} is 6 (the fourth units digit in the repeating sequence 8, 4, 2, 6).

When k is odd, that is, when $k = 2m + 1$ for positive integers m , we get the following

equivalent equations

$$\begin{aligned}
 2026k &= 2026 \times (2m + 1) \\
 &= 4052m + 2026 \\
 &= 4052m + 2024 + 2 \\
 &= 4 \times 1013m + 4 \times 506 + 2 \\
 &= 4(1013m + 506) + 2
 \end{aligned}$$

So, $2026k$ is 2 more than a multiple of 4 for all odd integers k .

Thus for all odd integers k , the units digit of 8^{2026k} is 4 (the second units digit in the repeating sequence 8, 4, 2, 6).

For 9^k , the consecutive units digits that repeat are 9, 1 with cycle length 2.

Thus when k is odd, the units digit of 9^k is 9, and when k is even, the units digit is 1.

For all integers k , $2026k$ is an even integer, and so the units digit of 9^{2026k} is 1 for all integers k .

Summarizing, we determined that the units digit of 8^{2026k} is 4 when k is odd, and is 6 when k is even. Also, the units digit of 9^{2026k} is 1 for all integers k .

Therefore, $8^{2026k} + 9^{2026k}$ has consecutive units digits $4 + 1 = 5$ and $6 + 1 = 7$ that repeat with cycle length 2.

Recall that $2^k + 3^k$ has consecutive units digits 5, 3, 5, 7 that repeat with cycle length 4. Thus, $2^k + 3^k$ and $8^{2026k} + 9^{2026k}$ have the same units digit, 5, for all odd values of k . Also, $2^k + 3^k$ and $8^{2026k} + 9^{2026k}$ have the same units digit, 7, for all values of k equal to a multiple of 4.

For $1 \leq k \leq 50$, there are 25 odd values of k and 12 values of k equal to a multiple of 4 (since $50 = 4 \times 12 + 2$), and so there are $25 + 12 = 37$ integers k for which $2^k + 3^k$ and $8^{2026k} + 9^{2026k}$ have the same units digit.

4. (a) In general, the x -coordinate of the midpoint of points A and B is equal to the average of the x -coordinates of points A and B , and its y -coordinate is equal to the average of the y -coordinates of points A and B .

For example, the midpoint of $(0, 4)$ and $(0, 16)$ is $\left(\frac{0+0}{2}, \frac{4+16}{2}\right) = (0, 10)$.

We determine the midpoint of each pair of distinct points from R in the table that follows.

	$(0, 0)$	$(0, 1)$	$(0, 4)$	$(0, 9)$	$(0, 16)$	$(0, 25)$
$(0, 1)$	$(0, 1/2)$					
$(0, 4)$	$(0, 2)$	$(0, 5/2)$				
$(0, 9)$	$(0, 9/2)$	$(0, 5)$	$(0, 13/2)$			
$(0, 16)$	$(0, 8)$	$(0, 17/2)$	$(0, 10)$	$(0, 25/2)$		
$(0, 25)$	$(0, 25/2)$	$(0, 13)$	$(0, 29/2)$	$(0, 17)$	$(0, 41/2)$	

In the table, there are $1 + 2 + 3 + 4 + 5 = 15$ midpoints. However, $(0, 25/2)$ is the midpoint of $(0, 16)$ and $(0, 9)$, as well as $(0, 25)$ and $(0, 0)$. Thus, there are 14 distinct midpoints.

- (b) From S , consider the distinct points $(i, i + 2)$ and $(j, j + 2)$ where i and j are integers, $i \neq j$, $1 \leq i \leq 20$ and $1 \leq j \leq 20$.

It then follows that each midpoint is of the form $\left(\frac{i+j}{2}, \frac{i+2+j+2}{2}\right) = \left(\frac{i+j}{2}, \frac{i+j}{2} + 2\right)$.

Since $i \neq j$, $1 \leq i \leq 20$ and $1 \leq j \leq 20$, then the maximum value of $i + j$ is $19 + 20 = 39$,

and the minimum value of $i + j$ is $1 + 2 = 3$.

Furthermore, all integer values of $i + j$ from 3 to 39 inclusive must occur at least once. (You should confirm this for yourself before reading on.)

Thus, the total number of distinct midpoints is $39 - 3 + 1 = 37$.

- (c) To minimize the number of distinct midpoints, the key idea is to choose a set of points T that lie in a straight line and are evenly spaced.

In this solution, we describe a set of points T for which the number of distinct midpoints is a minimum, determine the minimum m , and then show that fewer than m distinct midpoints is not possible for all possible choices for the set T .

Let T be the set of points of the form $(0, 2t)$, where t is an integer and $1 \leq t \leq 100$.

This set T contains exactly 100 points with distinct y -coordinates, as required.

From T , consider the distinct points $(0, 2k)$ and $(0, 2\ell)$ where k and ℓ are integers, $k \neq \ell$, $1 \leq k \leq 100$ and $1 \leq \ell \leq 100$.

It then follows that each midpoint is of the form $\left(0, \frac{2k + 2\ell}{2}\right) = (0, k + \ell)$.

Since $k \neq \ell$, $1 \leq k \leq 100$ and $1 \leq \ell \leq 100$, then the maximum value of $k + \ell$ is $99 + 100 = 199$, and the minimum value of $k + \ell$ is $1 + 2 = 3$.

All integer values of $k + \ell$ from 3 to 199 inclusive must occur at least once, and so the total number of distinct midpoints is $199 - 3 + 1 = 197$.

We have described a set of points T for which the number of distinct midpoints is $m = 197$. It remains to be shown why, for all possible choices for the set T , fewer than 197 midpoints is not possible.

Let the 100 distinct y -coordinates be $y_1 < y_2 < y_3 < y_4 < \cdots < y_{98} < y_{99} < y_{100}$.

The y -coordinate of the midpoint of the two points with y -coordinates y_n and y_{n+1} , where n is an integer and $1 \leq n \leq 99$, is $\frac{y_n + y_{n+1}}{2}$.

The y -coordinate of the midpoint of the two points with y -coordinates y_n and y_{n+2} , where n is an integer and $1 \leq n \leq 98$, is $\frac{y_n + y_{n+2}}{2}$.

Since $y_{n+1} < y_{n+2}$, then $\frac{y_n + y_{n+1}}{2} < \frac{y_n + y_{n+2}}{2}$ for integers n with $1 \leq n \leq 98$.

That is, $\frac{y_1 + y_2}{2} < \frac{y_1 + y_3}{2} < \frac{y_2 + y_3}{2} < \frac{y_2 + y_4}{2} < \cdots < \frac{y_{98} + y_{100}}{2} < \frac{y_{99} + y_{100}}{2}$, and so there are at least $m = 99 + 98 = 197$ distinct midpoints.

We have described a set of points T for which the number of distinct midpoints is $m = 197$ and we have shown that for all possible choices for the set T , $m \geq 197$.

Therefore, we have proven that $m = 197$.