The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2026 Euclid Contest

Tuesday, March 31, 2026
(in North America and South America)

Wednesday, April 1, 2026
(outside of North America and South America)

Solutions

1. (a) Multiplying both sides of the equation by $2 \cdot 3 = 6$, we obtain $6 \cdot \dfrac{2t}{3} + 6 \cdot \dfrac{3t}{2} = 6 \cdot 26$ or $4t + 9t = 156$.
Simplifying, we obtain $13t = 156$ and so $t = 12$.

   Alternatively, using a common denominator of $2 \cdot 3 = 6$, we obtain $\dfrac{2 \cdot 2t}{2 \cdot 3} + \dfrac{3 \cdot 3t}{3 \cdot 2} = 26$ or $\dfrac{4t}{6} + \dfrac{9t}{6} = 26$.
Simplifying, we obtain $\dfrac{13t}{6} = 26$ and so $13t = 6 \cdot 26$ or $t = 6 \cdot 2 = 12$.

   (b) Multiplying both sides of the equation by 8, we obtain $\dfrac{8(3+x)}{4} = 6 + x$ which simplifies to $2(3+x) = 6 + x$.
Simplifying further, we obtain $6 + 2x = 6 + x$ and so $x = 0$.

   Alternatively, splitting each fraction into two pieces, we obtain $\dfrac{3}{4} + \dfrac{x}{4} = \dfrac{6}{8} + \dfrac{x}{8}$.
Since $\dfrac{3}{4} = \dfrac{6}{8}$, we obtain $\dfrac{x}{4} = \dfrac{x}{8}$ and so $8x = 4x$ or $x = 0$.

   (c) Since $y > 0$, then $\sqrt{y^2} = y$.
From the given equation, we obtain $\sqrt{9 + 16 + 144} = \sqrt{9 + 16} + y$.
Thus, $\sqrt{169} = \sqrt{25} + y$ or $13 = 5 + y$ and so $y = 8$.

2. (a) We look for integers between 2026 and 2100 that have a sum of digits of 10.
Such integers are of the form $20xy$ for some digits x and y.
Since the sum of the digits is 10, then $2 + 0 + x + y = 10$ or $x + y = 8$.
For $20xy$ to be greater than 2026, we need $x \geq 2$.
If $x = 2$, then $y = 6$, which gives us the integer 2026.
To find the next greatest integer, we try $x = 3$, which gives $y = 5$. This gives us the integer 2035, which is the smallest integer $n > 2026$ with the desired property.

   (b) Consider the three-digit positive integers abc with product of digits equal to 9 (that is, $a \cdot b \cdot c = 9$).
Since $9 = 3^2$, then a, b and c are either 1, 1, 9 in some order or 1, 3, 3 in some order. (There are no other factors that can be used.)
The possible integers are 119, 191, 911, 133, 313, 331; there are 6 such integers.

   (c) Since the sum of x, $3x$ and $4y$ is equal to 48, then $x + 3x + 4y = 48$ and so $4x + 4y = 48$ or $x + y = 12$.
Since the average of x and y is equal to $3x$, then $\dfrac{x+y}{2} = 3x$ and so $x + y = 6x$ or $y = 5x$.
Subsituting $y = 5x$ into $x + y = 12$, we obtain $x + 5x = 12$ or $6x = 12$ and so $x = 2$.
Since $y = 5x$, then $y = 10$ and so $(x, y) = (2, 10)$.

3. (a) Using the prime factorization of each of the factors of the numerator, we see that

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 3^2 \cdot 2^3 \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 2^2 \cdot 3 = 2^6 \cdot 3^4 \cdot 5 \cdot 7$$

To find the smallest positive integer n for which $\dfrac{2^6 \cdot 3^4 \cdot 5 \cdot 7}{n}$ is a perfect cube, we look for the minimal set of prime divisors that we can remove (that is, divide out) from the numerator so that the number of times that each remaining prime occurs is a multiple of 3. This is because one way of characterizing a perfect cube is that each of its prime factors occurs in groups of 3.

To do this, we need to remove at least 1 factor of 3, at least 1 factor of 5, and at least 1 factor of 5. This means that $n \geq 3 \cdot 5 \cdot 7$.

If $n = 3 \cdot 5 \cdot 7 = 105$, then

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{n} = 2^6 \cdot 3^3 = (2^2 \cdot 3)^3$$

Since $n \geq 105$ and $n = 105$ gives a perfect cube, then the smallest possible n is $n = 105$.

(b) Since $3^{a+b} = 27$ and $3^3 = 27$, then $a + b = 3$.

Adding $a + b = 3$ to the equation $a - b = -5$, we obtain $2a = -2$ and so $a = -1$.

Since $b = 3 - a$, then $b = 4$ and so $(a, b) = (-1, 4)$.

(c) Since $P(10, 0)$ lies on the parabola with equation $y = -x^2 + 7x + c$, then $0 = -100 + 70 + c$ and so $c = 30$.

Thus, the parabola has equation $y = -x^2 + 7x + 30$ which can be factored to obtain $y = -(x - 10)(x + 3)$.

Since Q is the other x-intercept of the parabola, then Q has coordinates $(-3, 0)$.

Since R is the point where the parabola crosses the y-axis, we set $x = 0$ and obtain $y = 30$.

Thus, we want to find the area of the triangle with vertices $P(10, 0)$, $Q(-3, 0)$ and $R(0, 30)$.

We note that PQ is horizontal so can be treated as the base of the triangle. Also, $PQ = 10 - (-3) = 13$.

Point R is 30 units above PQ, so the height of the triangle relative to base PQ is 30.

Therefore, the area of $\triangle PQR$ is $\frac{1}{2} \cdot 13 \cdot 30 = 195$.

4. (a) Since the average of 31 temperatures was -20°C, then the sum of these 31 temperatures was $31 \cdot (-20°\text{C}) = -620°\text{C}$.

Since the average of 21 of these temperatures was -15°C, then the sum of these 21 temperatures was $21 \cdot (-15°\text{C}) = -315°\text{C}$.

This means that the sum of the other 10 temperatures was $-620°\text{C} - (-315°\text{C}) = -305°\text{C}$, and so the average of these other 10 temperatures was $\dfrac{-305°\text{C}}{10}$ or -30.5°C.

(b) Suppose that $MH = x$ km. This means that $HG = (10 - x)$ km.

Since McKayla runs on flat ground at 12 km/h, then the time that it takes her to run from M to H is $\dfrac{x \text{ km}}{12 \text{ km/h}}$ or $\dfrac{x}{12}$ h.

Since McKayla runs uphill at 10 km/h, then the time that it takes her to run from H to G is $\dfrac{(10 - x) \text{ km}}{10 \text{ km/h}}$ or $\dfrac{10 - x}{10}$ h.

Since it takes her 54 minutes to run from M to H to G, and 54 minutes is the same as $\frac{9}{10}$ h, then $\frac{x}{12}+\frac{10-x}{10}=\frac{9}{10}$.

Multiplying both sides of this equation by 120, we obtain $10x+12(10-x)=9\cdot 12$ and so $2x=12$ or $x=6$.

Therefore, to run from G to H to M, it takes McKayla

$$\frac{4\text{ km}}{15\text{ km/h}}+\frac{6\text{ km}}{12\text{ km/h}}=\frac{16}{60}\text{ h}+\frac{30}{60}\text{ h}=\frac{46}{60}\text{ h}$$

or 46 minutes.

5. (a) Suppose that n faces on D_2 have a 1 on them and $6-n$ faces have a 2 on them.

We make a chart to enumerate the possible totals when the two dice are rolled. The number rolled on D_1 is shown in the left column and the number rolled on D_2 is shown across the top row. Inside the chart, we track the sum and the number of times this sum could occur.

	1 (n times)	2 ($6-n$ times)
1	2 (n times)	3 ($6-n$ times)
2	3 (n times)	4 ($6-n$ times)
3	4 (n times)	5 ($6-n$ times)
4	5 (n times)	6 ($6-n$ times)
5	6 (n times)	7 ($6-n$ times)
6	7 (n times)	8 ($6-n$ times)

Of these sums, 2, 3, 5, and 7 are prime.

These occur a total of $n+(6-n)+n+(6-n)+n+(6-n)+n=18+n$ times out of the possible $6\times 6=36$ outcomes from rolling the two dice together.

Since the probability of having a prime sum is $\frac{23}{36}$, then 23 of the 36 outcomes give a prime sum, and so $18+n=23$ or $n=5$.

(b) Since $ABCD$ is a square and AB is horizontal, then CD is parallel to AB and so is also horizontal.

Since C and D are on a parabola and CD is horizontal, then C and D are equidistant from the axis of symmetry.

Since the parabola has equation $y=x^2-4$, its x-intercepts are 2 and -2 and so its axis of symmetry has equation $x=0$.

Thus, we can say that C and D have x-coordinates s and $-s$, respectively, for some $s>0$.

This means that A has coordinates $(-s,0)$ and B has coordinates $(s,0)$.

This means that the side length of square $ABCD$ is $s-(-s)=2s$.

Since the height and width of $ABCD$ are equal, then C has coordinates $(s,-2s)$ and D has coordinates $(-s,-2s)$.

Since C lies on the parabola with equation $y=x^2-4$, then $-2s=s^2-4$ and so $s^2+2s-4=0$.

By the quadratic formula,

$$s=\frac{-2\pm\sqrt{2^2-4(1)(-4)}}{2}=\frac{-2\pm\sqrt{20}}{2}=\frac{-2\pm 2\sqrt{5}}{2}=-1\pm\sqrt{5}$$

Since $s>0$, then $s=-1+\sqrt{5}$.

This means that the area of square $ABCD$ is equal to $(2s)^2$ which equals $(-2+2\sqrt{5})^2$. Expanding and simplifying, we obtain $4+20-8\sqrt{5} = 24-8\sqrt{5} = 24-\sqrt{320}$.

6. (a) Xander's rope is 10 m long.

Since Yasmin's rope is $n\%$ longer than Xander's rope, then the length of Yasmin's rope is $10\left(1+\dfrac{n}{100}\right)$ m.

Since Zhe's rope is $(2n)\%$ longer than Yasmin's rope, then the length of Zhe's rope is $10\left(1+\dfrac{n}{100}\right)\left(1+\dfrac{2n}{100}\right)$ m.

Since Zhe's rope is $(3.14n)\%$ longer than Xander's rope, then the length of Zhe's rope can also be written as $10\left(1+\dfrac{3.14n}{100}\right)$ m.

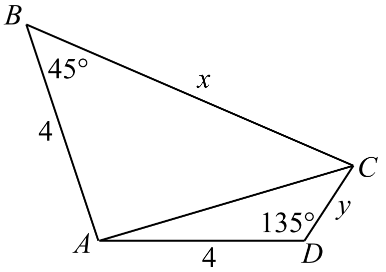
Therefore,

$$\begin{aligned} 10\left(1+\frac{n}{100}\right)\left(1+\frac{2n}{100}\right) &= 10\left(1+\frac{3.14n}{100}\right) \\ \left(1+\frac{n}{100}\right)\left(1+\frac{2n}{100}\right) &= \left(1+\frac{3.14n}{100}\right) \\ (100+n)(100+2n) &= 100(100+3.14n) \qquad \text{(multiplying by } 100\cdot 100\text{)} \\ 10000+300n+2n^2 &= 10000+314n \\ 2n^2-14n &= 0 \\ 2n(n-7) &= 0 \end{aligned}$$

Since $n>0$, then it must be the case that $n=7$.

(b) *Solution 1*

Let $BC = x$ and $CD = y$. Join A to C.



Using the cosine law in $\triangle ABC$, we obtain

$$\begin{aligned} AC^2 &= AB^2+BC^2-2(AB)(BC)\cos(\angle ABC) \\ &= 16+x^2-8x\cos(45^\circ) \\ &= 16+x^2-8x(\tfrac{1}{\sqrt{2}}) \\ &= 16+x^2-4\sqrt{2}x \end{aligned}$$

Using the cosine law in $\triangle ADC$, we obtain

$$\begin{aligned} AC^2 &= AD^2 + DC^2 - 2(AD)(DC)\cos(\angle ADC) \\ &= 16 + y^2 - 8y\cos(135^\circ) \\ &= 16 + y^2 - 8y(-\tfrac{1}{\sqrt{2}}) \\ &= 16 + y^2 + 4\sqrt{2}y \end{aligned}$$

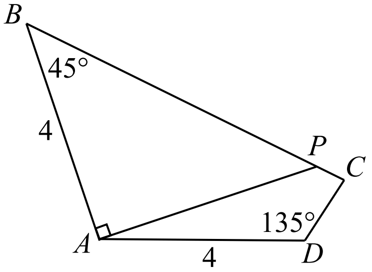
Equating expressions for AC^2, we obtain

$$\begin{aligned} 16 + x^2 - 4\sqrt{2}x &= 16 + y^2 + 4\sqrt{2}y \\ x^2 - y^2 - 4\sqrt{2}x - 4\sqrt{2}y &= 0 \\ (x+y)(x-y) - 4\sqrt{2}(x+y) &= 0 \\ (x+y)(x-y-4\sqrt{2}) &= 0 \end{aligned}$$

Since $x > 0$ and $y > 0$, then $x + y > 0$. Thus, $x - y - 4\sqrt{2} = 0$ and so $BC - CD = x - y = 4\sqrt{2}$.

Solution 2

Let point P be on BC so that AP is perpendicular to AB.



To see why P is on BC (and not some extension of BC) first observe that isosceles $\triangle BAD$ has $\angle ADB = \angle ABD < \angle ABC = 45^\circ$, so

$$\angle BAD = 180^\circ - \angle ADB - \angle ABD > 180^\circ - 45^\circ - 45^\circ = 90^\circ$$

Therefore, $\angle BAD$ is obtuse.

Now suppose P were on some extension of BC. Since $\angle BAD$ is obtuse and $\angle BAP = 90^\circ$, AP must intersect CD at some point M, and so $AM < AP$. However, $AP = AB = 4$ since $\triangle BAP$ is is a right-isosceles triangle, which means in $\triangle AMD$, we have that AM is not the longest side while it is opposite obtuse $\angle ADM$. This is impossible, so we conclude that P must be on BC.

It was mentioned above that $\triangle BAP$ is right-angled and isosceles, with $AP = AB = 4$ which means that $BP = \sqrt{2}AB = 4\sqrt{2}$.
Since $\angle BPA = 45^\circ$ and BPC is a straight angle, then $\angle CPA = 180^\circ - \angle BPA = 135^\circ$. Therefore, $\angle APC = \angle ADC$.
Since $AP = AD = 4$, then $\triangle APD$ is isosceles, and so $\angle APD = \angle ADP$. Then

$$\angle CPD = \angle APC - \angle APD = \angle ADC - \angle ADP = \angle CDP$$

Since $\angle CPD = \angle CDP$, then $\triangle CPD$ is isosceles, and so $CD = CP$. Thus, $BC - CD = BC - CP = BP = 4\sqrt{2}$.

7. (a) We make a table to track the data which we are given:

	Yellow	White
Roses		
Carnations		

Since half of the yellow flowers are roses, we suppose that there are $3n$ yellow roses and $3n$ yellow carnations for some real number n. (We will see why choosing $3n$ and $3n$ instead of n and n is useful here.)
Since $\frac{1}{4}$ of the roses are yellow, then $\frac{3}{4}$ of the roses are white, which means that there are $3 \cdot 3n = 9n$ white roses.
We can now update the table:

	Yellow	White
Roses	$3n$	$9n$
Carnations	$3n$	

Since $\frac{1}{4}$ of all of the flowers are white carnations, then $\frac{3}{4}$ of the flowers are accounted for by the $3n + 9n + 3n = 15n$ flowers accounted for so far.
Thus, there are $\frac{1}{3} \cdot 15n = 5n$ white carnations. (If we had chosen n instead of $3n$ initially, we would get a fractional coefficient here.)
Therefore, there are $15n + 5n = 20n$ flowers in total, of which $3n$ are yellow roses.
This means that $\frac{3}{20}$ of the flowers are yellow roses.

(b) Using the definition of the function f which is

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 + f(\log_2 x) & \text{if } x > 1 \end{cases}$$

then $f(n) = 4$ exactly when $n > 1$ and $f(\log_2 n) = 3$. (Note that the first alternative in the definition is not the relevant one.)
Continuing to use the definition, we have the following:

- $f(\log_2 n) = 3$ exactly when $\log_2 n > 1$ and $f(\log_2(\log_2 n)) = 2$.
  (We could think of $t = \log_2 n$ and so $f(t) = 3$ exactly when $t > 1$ and $f(\log_2 t) = 2$.)
- $f(\log_2(\log_2 n)) = 2$ exactly when $\log_2(\log_2 n) > 1$ and $f(\log_2(\log_2(\log_2 n))) = 1$.
- $f(\log_2(\log_2(\log_2 n))) = 1$ exactly when $\log_2(\log_2(\log_2 n)) > 1$ and $f(\log_2(\log_2(\log_2(\log_2 n)))) = 0$.
- $f(\log_2(\log_2(\log_2(\log_2 n)))) = 0$ exactly when $0 < \log_2(\log_2(\log_2(\log_2 n))) \leq 1$.

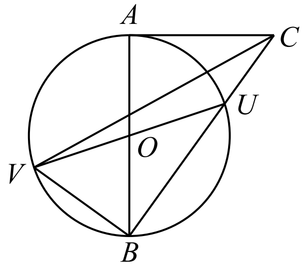
Therefore, we know so far that $f(n) = 4$ exactly when $0 < \log_2(\log_2(\log_2(\log_2 n))) \leq 1$.
Since the logarithm is a strictly increasing function, then $0 < \log_2(\log_2(\log_2(\log_2 n))) \leq 1$ exactly when $2^0 < \log_2(\log_2(\log_2 n)) \leq 2^1$ or $1 < \log_2(\log_2(\log_2 n)) \leq 2$.
Continuing to invert the logarithm functions,

- $1 < \log_2(\log_2(\log_2 n)) \leq 2$ exactly when $2^1 < \log_2(\log_2 n) \leq 2^2$ or $2 < \log_2(\log_2 n) \leq 4$.
- $2 < \log_2(\log_2 n) \leq 4$ exactly when $2^2 < \log_2 n \leq 2^4$ or $4 < \log_2 n \leq 16$.
- $4 < \log_2 n \leq 16$ exactly when $2^4 < n \leq 2^{16}$ or $16 < n \leq 65536$.

In other words, $f(n) = 4$ exactly when $16 < n \leq 65536$. Since n is an integer, then $f(n) = 4$ exactly when $17 \leq n \leq 65536$.
There are thus $65536 - 16 = 65520$ positive integers n for which $f(n) = 4$.

8. (a) *Solution 1*

Since the circle has radius 2, then $AO = OB = VO = OU = 2$ and so $AB = VU = 4$.



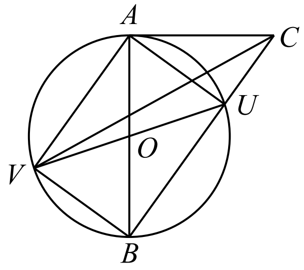
Since AC is tangent to the circle at A, then $\angle CAB = 90^\circ$.
Since UV is a diameter of the circle, then $\angle UBV = 90^\circ$.
Since $BU = 2UC$, we let $UC = t$ and $BU = 2t$ for some $t > 0$; thus, $BC = 3t$.
By the Secant-Tangent Theorem, $CA^2 = CU \cdot CB = t \cdot 3t = 3t^2$.
By the Pythagorean Theorem in $\triangle ABC$, $BC^2 = AB^2 + AC^2$ or $(3t)^2 = 4^2 + 3t^2$ from which we obtain $6t^2 = 16$ or $t = \sqrt{\frac{8}{3}}$ since $t > 0$.
Now in $\triangle CUV$, we consider $CU = t = \sqrt{\frac{8}{3}}$ as its base and VB as its height. (Note that VB is perpendicular to the extension of the base.)
By the Pythagorean Theorem in $\triangle VBU$, we have

$$VB = \sqrt{VU^2 - BU^2} = \sqrt{4^2 - (2t)^2} = \sqrt{16 - 4t^2} = \sqrt{16 - 4 \cdot \frac{8}{3}} = \sqrt{\frac{16}{3}}$$

since $VB > 0$.
Finally, the area of $\triangle CUV$ is equal to $\frac{1}{2} \cdot \sqrt{\frac{8}{3}} \cdot \sqrt{\frac{16}{3}} = \frac{1}{2}\sqrt{\frac{128}{9}} = \frac{1}{2} \cdot \frac{8}{3} \cdot \sqrt{2} = \frac{4\sqrt{2}}{3}$.

Solution 2

Join A to U, A to V, and B to V.
Since AB is a diameter of the circle, then $\angle AUB = \angle AVB = 90^\circ$. Similarly, since UV is a diameter, then $\angle VBU = 90^\circ$.



Since quadrilateral $AVBU$ has four right angles, it is a rectangle.
We consider $\triangle CUV$ as having base CU and exterior height VB. (Note that BV is perpendicular to CU extended since $AVBU$ is a rectangle.)
Since $AVBU$ is a rectangle, $BV = AU$ and so the area of $\triangle CUV$ is equal to $\dfrac{1}{2} \cdot UC \cdot AU$.
Let $UC = t$. Thus, $BU = 2UC = 2t$ and $BC = BU + UC = 3t$.
Since BA is a diameter of the circle and AC is tangent to the circle, then $\angle BAC = 90^\circ$.
Consider the three triangles $\triangle BUA$, $\triangle BAC$ and $\triangle AUC$.
These are right-angled at U, A and U, respectively.
Since $\triangle BUA$ and $\triangle BAC$ share a common angle at B, they are similar. This means that

$\dfrac{AB}{BU} = \dfrac{BC}{AB}$.

Since $\triangle BAC$ and $\triangle AUC$ share a common angle at C, they are similar. This in turn means that $\triangle BUA$ is similar to $\triangle AUC$ and so $\dfrac{UC}{AU} = \dfrac{AU}{BU}$.

Since the radius of the circle is 2, the diameter is 4 and so $AB = 4$.

Since $\dfrac{AB}{BU} = \dfrac{BC}{AB}$, then $\dfrac{4}{2t} = \dfrac{3t}{4}$ and so $6t^2 = 16$ or $t^2 = \dfrac{8}{3}$.

Since $t > 0$, this gives $t = \sqrt{\dfrac{8}{3}}$.

Since $\dfrac{UC}{AU} = \dfrac{AU}{BU}$, then $\dfrac{t}{AU} = \dfrac{AU}{2t}$ and so $AU^2 = 2t^2 = \dfrac{16}{3}$.

Since $AU > 0$, then $AU = \sqrt{\dfrac{16}{3}}$.

Therefore, the area of $\triangle CUV$ is

$$\frac{1}{2} \cdot UC \cdot AU = \frac{1}{2} \cdot t \cdot AU = \frac{1}{2} \cdot \sqrt{\frac{8}{3}} \cdot \sqrt{\frac{16}{3}} = \frac{\sqrt{128}}{2 \cdot 3} = \frac{8\sqrt{2}}{2 \cdot 6} = \frac{4\sqrt{2}}{3}$$

Solution 3

That $UC = t$ and so $BU = 2UC = 2t$ and $BC = BU + UC = 3t$.

Since the radius of the circle is 2, then $OB = OU = 2$.

Suppose that $\angle OBU = \theta$.

Since $\triangle OBU$ is isosceles with $OB = OU$, then $\angle OUB = \angle OBU = \theta$ and $\angle BOU = 180^\circ - 2\theta$.

Using the sine law in $\triangle OBU$, we obtain $\dfrac{\sin(\angle OUB)}{OU} = \dfrac{\sin(\angle BOU)}{BU}$.

Thus, $\dfrac{\sin\theta}{2} = \dfrac{\sin(180^\circ - 2\theta)}{2t}$ which gives $t\sin\theta = \sin(180^\circ - 2\theta)$.

Note that $\sin(180^\circ - 2\theta) = \sin 2\theta = 2\sin\theta\cos\theta$.

Thus, $t\sin\theta = 2\sin\theta\cos\theta$ or $t = 2\cos\theta$ since $0^\circ < \theta < 180^\circ$ which means that $\sin\theta \neq 0$.

Next, AB is a diameter of the circle and AC is a tangent to the circle, which means that $\angle BAC = 90^\circ$.

Therefore, $\triangle BAC$ gives $\cos\theta = \dfrac{AB}{BC} = \dfrac{4}{3t}$.

Substituting, we obtain $t = \dfrac{8}{3t}$ and so $t^2 = \dfrac{8}{3}$.

We can now proceed as in Solution 1 or Solution 2 to calculate the area of $\triangle CUV$.

(b) Let k be a positive integer.

Suppose that one of these triangles has side lengths a, k and c.

Since the perimeter must be $3k$, then $a + k + c = 3k$ or $a + c = 2k$.

Suppose that $a = k - t$ for some integer $t \geq 0$; thus, $c = 2k - a = k + t$. (Either a or c is at most c, so we assume that $a \leq k$.)

Therefore, the side lengths are $k - t$, k and $k + t$ for some integer $t \geq 0$. Note that $k - t \leq k \leq k + t$.

For these to be the side lengths of a triangle, the Triangle Inequality must be satisfied.

Since we already know the relative sizes of the side lengths, the existence of a triangle with these side lengths is equivalent to determining integers $k > 0$ and $t \geq 0$ that satisfy the inequality $k + t < (k - t) + k$, which simplifies to $2t < k$ or $t < \frac{1}{2}k$.

Next, the triangle must be obtuse.

Now, $\triangle ABC$ with side lengths $BC = a$ and $AC = b$ and $AB = c$ with $a \leq b \leq c$ is obtuse exactly when $a^2 + b^2 < c^2$:

> We can see this by noting that the angle with greatest measure is opposite the longest side length c, and so the angle with greatest measure is $\angle ACB$.
> Using the cosine law, we note that
>
> $$c^2 = a^2 + b^2 - 2ab\cos(\angle ACB)$$
>
> Since $a > 0$ and $b > 0$, then $\cos(\angle ACB) < 0$ (that is, $\angle ACB$ is obtuse) exactly when $-2ab\cos(\angle ACB) > 0$ which is exactly when $c^2 > a^2 + b^2$.

Using this result, the triangle with side lengths $k - t$, k, $k + t$ is obtuse exactly when $(k - t)^2 + k^2 < (k + t)^2$.
Expanding and simplifying, this inequality is true exactly when

$$k^2 - 2kt + t^2 + k^2 < k^2 + 2kt + t^2$$

or $k^2 < 4kt$ or $t > \frac{1}{4}k$ (since $k > 0$).
Thus, for a given (fixed) positive integer k, the requirements for the triangle are equivalent to the integer t satisfying the inequality $\frac{1}{4}k < t < \frac{1}{2}k$.
This means that we need to find all positive integers k for which there are exactly 100 integers t with $\frac{1}{4}k < t < \frac{1}{2}k$.

To do this, we note that every positive integer k can be written in the form $k = 4q$ or $k = 4q + 1$ or $k = 4q + 2$ or $k = 4q + 3$ for some non-negative integer q.

If $k = 4q$, the inequality becomes $q < t < 2q$.
Since q is a non-negative integer, we can rewrite this inequality as $q + 1 \leq t \leq 2q - 1$.
There are $(2q - 1) - (q + 1) + 1 = q - 1$ integers in this interval.
For there to be 100 integers in this interval, $q = 101$ and so $k = 404$.

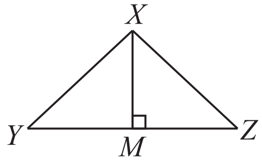
If $k = 4q + 1$, the inequality becomes $q + \frac{1}{4} < t < 2q + \frac{1}{2}$.
Since q is a non-negative integer, we can rewrite this inequality as $q + 1 \leq t \leq 2q$.
There are $2q - (q + 1) + 1 = q$ integers in this interval.
For there to be 100 integers in this interval, $q = 100$ and so $k = 401$.

If $k = 4q + 2$, the inequality becomes $q + \frac{1}{2} < t < 2q + 1$.
Since q is a non-negative integer, we can rewrite this inequality as $q + 1 \leq t \leq 2q$.
There are $2q - (q + 1) + 1 = q$ integers in this interval.
For there to be 100 integers in this interval, $q = 100$ and so $k = 402$.

If $k = 4q + 3$, the inequality becomes $q + \frac{3}{4} < t < 2q + \frac{3}{2}$.
Since q is a non-negative integer, we can rewrite this inequality as $q + 1 \leq t \leq 2q + 1$.
There are $2q + 1 - (q + 1) + 1 = q + 1$ integers in this interval.
For there to be 100 integers in this interval, $q = 99$ and so $k = 399$.

Therefore, the positive integers k that satisfy the conditions are $k = 399, 401, 402, 404$.

9. (a) Suppose that $XY = XZ = a$. This means that the triangle has side lengths a, a, b.
Since the perimeter of the triangle is 32, then $a + a + b = 32$ and so $2a = 32 - b$ or $a = 16 - \frac{1}{2}b$.
Suppose that M is the midpoint of YZ; thus, $YM = MZ = \frac{1}{2}b$.



Since M is the midpoint of YZ and $\triangle XYZ$ is isosceles, then XM is perpendicular to YZ.
By the Pythagorean Theorem,

$$XM = \sqrt{XY^2 - YM^2} = \sqrt{a^2 - \left(\tfrac{1}{2}b\right)^2} = \sqrt{a^2 - \tfrac{1}{4}b^2}$$

since $XM > 0$.
Since the area of $\triangle XYZ$ is 40, then $\frac{1}{2} \cdot YZ \cdot XM = 40$.
Combining these pieces of information, we obtain the following equivalent equations:

$$\begin{aligned}
\tfrac{1}{2} \cdot YZ \cdot XM &= 40 \\
b\sqrt{a^2 - \tfrac{1}{4}b^2} &= 80 \\
b^2\left(\left(16 - \tfrac{1}{2}b\right)^2 - \tfrac{1}{4}b^2\right) &= 80^2 \\
b^2\left(256 - 16b + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\right) &= 6400 \\
b^2(256 - 16b) &= 6400 \\
b^2(16 - b) &= 400 \\
0 &= b^3 - 16b^2 + 400
\end{aligned}$$

Therefore, if we set $f(x) = x^3 - 16x^2 + 400$, then the polynomial f has the required properties.

(b) Suppose that $\triangle XYZ$ has $XY = XZ = a$, $YZ = b$, perimeter P and area A.

Generalizing our work from (a), we obtain $2a + b = P$ or $a = \frac{1}{2}P - \frac{1}{2}b$, as well as $\frac{1}{2}b\sqrt{a^2 - \frac{1}{4}b^2} = A$.

Using these equations, we obtain the following equivalent equations:

$$\begin{aligned} b\sqrt{a^2 - \tfrac{1}{4}b^2} &= 2A \\ b^2\left(\left(\tfrac{1}{2}P - \tfrac{1}{2}b\right)^2 - \tfrac{1}{4}b^2\right) &= 4A^2 \\ b^2\left(\tfrac{1}{4}P^2 - \tfrac{1}{2}Pb + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\right) &= 4A^2 \\ b^2\left(\tfrac{1}{4}P^2 - \tfrac{1}{2}Pb\right) &= 4A^2 \\ b^2(P^2 - 2Pb) &= 16A^2 \\ 0 &= 2Pb^3 - P^2b^2 + 16A^2 \\ 0 &= b^3 - \frac{P}{2}b^2 + \frac{8A^2}{P} \end{aligned}$$

Therefore, b is a root of the cubic polynomial $f(x) = x^3 - \dfrac{P}{2}x^2 + \dfrac{8A^2}{P}$.

There is nothing special about "b" in the work above, so in fact any positive real root of this cubic polynomial will be the length of YZ in an isosceles triangle $\triangle XYZ$ with $XY = XZ$, area A and perimeter P.

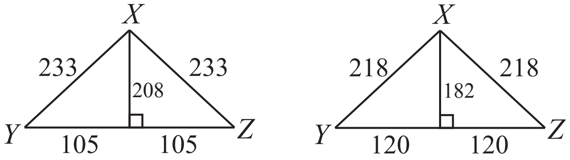
Now, the cubic polynomial $f(x) = x^3 - \dfrac{P}{2}x^2 + \dfrac{8A^2}{P}$ has constant term $\dfrac{8A^2}{P}$ which is positive.

This means that the product of the roots of the cubic polynomial $f(x)$ is $-\dfrac{8A^2}{P}$, which is negative.

This means that f cannot have 3 roots which are all positive real numbers, since their product would be positive.

Thus, f has at most 2 positive real roots, which means that there exist at most 2 non-congruent isosceles triangles with area A and perimeter P.

(c) One pair of positive integers A and P that have this property are $A = 21840$ and $P = 676$. The triangles below are non-congruent, isosceles and have side lengths less than 300. They both have perimeter $P = 676$. Additionally, they both have area $A = 21840$ because $\frac{1}{2} \cdot 210 \cdot 208 = 21840$ and $\frac{1}{2} \cdot 240 \cdot 182 = 21840$.
Finally, we note also that 210 is divisible by 7 and 240 is not divisible by 7.



How could we find these integers A and P in a systematic way?

From (b), we want to find A and P for which the cubic polynomial $f(x) = x^3 - \frac{P}{2}x^2 + \frac{8A^2}{P}$ has two distinct positive integer roots.

Suppose that the polynomial $f(x) = x^3 - \frac{P}{2}x^2 + \frac{8A^2}{P}$ has roots a, b and c, and that a and b are positive integers.

Using knowledge about the connection between the sums and products of the roots of a cubic polynomial and the coefficients of the polynomial, the following three equations are true:

$$a + b + c = \frac{P}{2} \qquad ab + bc + ac = 0 \qquad abc = -\frac{8A^2}{P}$$

From the second equation, $ab + c(a + b) = 0$ from which we obtain $c = -\frac{ab}{a+b}$. (Since $a > 0$ and $b > 0$, then $a + b > 0$.)

From the first equation, $a + b + c = a + b - \frac{ab}{a+b} = \frac{P}{2}$ and so

$$P = 2a + 2b - \frac{2ab}{a+b} = \frac{2a^2 + 4ab + 2b^2 - 2ab}{a+b} = \frac{2a^2 + 2ab + 2b^2}{a+b}$$

From the third equation,

$$8A^2 = -Pabc = -\left(\frac{2a^2 + 2ab + 2b^2}{a+b}\right) \cdot ab \cdot \left(-\frac{ab}{a+b}\right) = \frac{a^2b^2(2a^2 + 2ab + 2b^2)}{(a+b)^2}$$

and so

$$A = \frac{ab\sqrt{a^2 + ab + b^2}}{2(a+b)}$$

Given that a and b are positive integers, we need at the very least for $\sqrt{a^2 + ab + b^2}$ to be an integer, otherwise the value of A would be irrational. (This doesn't guarantee us that A is an integer, but is a place to start.)

From the conditions in (c), we want one of a and b to be divisible by 7. (Recall that a and b are roots of the polynomial $f(x)$, whose roots are possible lengths of YZ.)

We start by trying $a = 7$ and see if we can find another integer b for which $a^2 + ab + b^2$ is a perfect square.

After some trial and error, we can find that when $a = 7$ and $b = 8$, we get

$$a^2 + ab + b^2 = 49 + 56 + 64 = 169 = 13^2$$

When $a = 7$ and $b = 8$, we obtain

$$P = \frac{2a^2 + 2ab + 2b^2}{a+b} = \frac{338}{15} \qquad A = \frac{ab\sqrt{a^2 + ab + b^2}}{2(a+b)} = \frac{56 \cdot 13}{2 \cdot 15} = \frac{364}{15}$$

Here, P and A are not integers.
If we take $a = 7$ and $b = 8$ and scale the side lengths of the corresponding triangles up by a factor of 15, the perimeter is scaled up by a factor of 15 and the area of the triangle is scaled up by a factor of $15^2 = 225$.
Thus, we would have $a = 105$ and $b = 120$ and

$$P = \frac{338}{15} \cdot 15 = 338 \qquad A = \frac{364}{15} \cdot 15^2 = 5460$$

While this looks like it should work, if the other equal side lengths of the isosceles triangle with $a = 105$ are x and x, we obtain $105 + 2x = 338$, giving $x = 116.5$, which is not an integer.
To resolve this issue, we scale the side lengths of the triangles up by an additional factor of 2 to obtain $a = 210$ and $b = 240$ and

$$P = 338 \cdot 2 = 676 \qquad A = 5460 \cdot 4 = 21840$$

This gives us triangles of side lengths 210, 233, 233 and 240, 218, 218. These triangles have positive integer side lengths less than 300, have equal perimeters of 676, and have exactly one of the non-equal side lengths divisible by 7.
Noting that $\sqrt{233^2 - 105^2} = 208$ and $\sqrt{218^2 - 120^2} = 182$ gives us the heights of the triangles as seen in the diagrams above and confirms that the areas of these triangles are equal to 21840.
As a final check, we note that $\frac{P}{2} = 338$ and that $\frac{8A^2}{P} = 5644800$ and that

$$\begin{aligned} f(x) &= x^3 - 338x^2 + 5644800 \\ &= (x - 210)(x^2 - 128x - 26880) \\ &= (x - 210)(x - 240)(x + 112) \end{aligned}$$

and so the polynomial $f(x)$ does have the correct roots.

10. (a) Suppose that the arrangement a_1, a_2, a_3, a_4, a_5 of 1, 2, 3, 4, 5 has an internal peak at position 3 and has no other internal peaks.
Given that $a_3 > a_2$ and $a_3 > a_4$, then $a_3 \geq 3$ which means that $a_3 = 3$ or $a_3 = 4$ or $a_3 = 5$.

Case 1: $a_3 = 3$
Neither a_2 nor a_4 can equal 4 or 5.
Therefore, a_1 and a_5 are equal to 4 and 5 in some order.
This means that a_2 and a_4 are equal to 1 and 2 in some order.
Each of these 4 possible arrangements has an internal peak at position 3 and at no other position:

$$4,1,3,2,5 \qquad 4,2,3,1,5 \qquad 5,1,3,2,4 \qquad 5,2,3,1,4$$

There are 4 arrangements in this case.

Case 2: $a_3 = 4$
Neither a_2 nor a_4 can equal 5.
Therefore, either $a_1 = 5$ or $a_5 = 5$.
The remaining 3 entries can equal 1, 2, 3 in any of the 6 possible arrangements.
The following 6 arrangements show this for $a_1 = 5$:

$$5,1,4,2,3 \qquad 5,1,4,3,2 \qquad 5,2,4,1,3 \qquad 5,2,4,3,1 \qquad 5,3,4,1,2 \qquad 5,3,4,2,1$$

These 6 arrangements can be written in reverse to obtain the arrangements with $a_5 = 5$.
There are 12 arrangements in this case.

Case 3: $a_3 = 5$
The entries a_1 and a_2 will be both less than 5, since 5 is the largest number in the list.
Additionally, whether $a_1 < a_2 < 5$ or $a_1 > a_2$ and $a_2 < 5$, there is not an internal peak at position 2. (There cannot be an internal peak at position 1.)
This means that any 2 of the remaining 4 numbers can be at a_1 and a_2 and in any order.
The same is true for a_4 and a_5.
Therefore, there are 4 possible entries for a_1, and then 3 possible entries for a_2, and then 2 possible entries for a_4, and then 1 possible entry for a_5.
This means that there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible arrangements in this case.

In total, there are $4 + 12 + 24 = 40$ such arrangements.

(b) Suppose that $n \geq 5$ and that the arrangement a_1, a_2, …, a_{n-1}, a_n of $1, 2, \ldots, n-1, n$ has an internal peak at position 2 and has no other internal peaks.
Given that $a_2 > a_1$ and $a_2 > a_3$, then $a_2 \geq 3$ which means that $3 \leq a_2 \leq n$.
Since the portion of the arrangement from a_2 to a_n has no internal peaks and has $a_2 > a_3$, then the terms from a_2 to a_n are either strictly decreasing, or they are strictly decreasing followed by strictly increasing. In particular, their direction (decreasing/increasing) cannot turn again from increasing to decreasing (that is, their direction changes at most once).
Among other things, this means that any terms in the sequence that are greater than a_2 must occur at the rightmost end of the sequence and must be listed there in increasing order. This is because terms that are greater than a_2 cannot occur at a_1 or in the decreasing segment starting at a_2; since they are thus an increasing segment, they must come at the end and in increasing order.

Consider the case that $a_2 = n$.
Here, there are $n - 1$ possibilities for a_1 (since every remaining value is less than n) and the remaining $n - 2$ numbers are put in positions a_3 through a_n.
Suppose that $a_1 = n - 1$ and that the numbers $1, 2, \ldots, n-3, n-2$ are put in positions

a_3 through a_n.

If a_1 were a different number less than $n-1$, the argument that follows can be adapated by re-labelling the numbers placed in a_3 through a_n as $1, 2, \ldots, n-3, n-2$ in increasing order. It is only their relative value that matters here, not their relationship with the value of a_1.

Since $a_2 = n$ and the terms from a_3 to a_n are strictly decreasing, or strictly decreasing and then strictly increasing, the largest remaining number must be at one of the two ends (either a_3 or a_n). Thus, either $a_3 = n-2$ or $a_n = n-2$. In other words, there are 2 possible positions for $n-2$.

Once $n-2$ is placed, there are then 2 possible positions for $n-3$: either at the leftmost or the rightmost end of the remaining block of $n-3$ terms. If $n-3$ were in the middle of this block somewhere, it would have two neighbours smaller than it, and so would give an internal peak.

We continue to place $n-4$ through 2 in decreasing order. Each of these $n-5$ numbers has 2 possible positions. This then leaves 1 possible position for 1 as the only remaining term.

Pulling this together, there are $n-1$ possible values for a_1, 2 possible positions for each of $n-3$ numbers, and 1 possible position for the final unplaced number.

In total, there are $(n-1) \cdot 2^{n-3}$ arrangements with $a_2 = n$.

Consider next the case that $a_2 = n-1$.

Here, there are $n-2$ possibilities for a_1 (each number from 1 through $n-2$ can go there).

The number n must go in position a_n because it is larger than $a_2 = n-1$ and there are no other internal peaks.

The remaining $n-3$ numbers must be terms a_3 through a_{n-1}. As in the case that $a_2 = n$, we place these numbers one at a time starting with the largest remaining number (which can go in 2 places), the next largest remaining number (2 places), and so on. There are $n-4$ numbers each with 2 possible positions. The final number has only 1 possible position.

In this case, there are $(n-2) \cdot 2^{n-4}$ arrangements.

In general, suppose that $a_2 = n-k$ where $0 \leq k \leq n-3$.

Here, there are $n-k-1$ possibilities for a_1 (each number from 1 through $n-k-1$ can go there).

The numbers n, $n-1$, …, $n-k+1$ must go in positions a_n, a_{n-1}, …, a_{n-k+1}, respectively, because they are larger than $a_2 = n-k$ and there are no other internal peaks.

The remaining $n-k-2$ numbers must be terms a_3 through a_{n-k}. Again, there are $n-k-3$ numbers each with 2 possible positions. The final number has only 1 possible position.

In this case, there are $(n-k-1) \cdot 2^{n-k-3}$ arrangements.

In total, the number of arrangements is

$$(n-1) \cdot 2^{n-3} + (n-2) \cdot 2^{n-4} + (n-3) \cdot 2^{n-5} + \cdots + 4 \cdot 2^2 + 3 \cdot 2^1 + 2 \cdot 2^0$$

The last three terms come from $k = n-5$, $k = n-4$, and $k = n-3$. We note that when $a_2 = 3$ (corresponding to $k = n-3$) there are 2 choices for a_1 (either 2 or 1), the numbers $n, n-1, \ldots, 5, 4$ are filled in from the right, and a_3 is the remaining number, which is either 1 or 2.

Manipulating this sum, we obtain

$$\begin{aligned}
&(n-1)\cdot 2^{n-3}+(n-2)\cdot 2^{n-4}+(n-3)\cdot 2^{n-5}+\cdots+4\cdot 2^2+3\cdot 2^1+2\cdot 2^0\\
&\quad= 2\cdot 2^0+3\cdot 2^1+\cdots+(n-2)\cdot 2^{n-4}+(n-1)\cdot 2^{n-3}\\
&\quad= \tfrac{1}{4}\left(2\cdot 2^2+3\cdot 2^3+\cdots+(n-2)\cdot 2^{n-2}+(n-1)\cdot 2^{n-1}\right)\quad\text{(multiplying and dividing by 4)}\\
&\quad= \tfrac{1}{4}\left(0\cdot 2^0+1\cdot 2^1+2\cdot 2^2+3\cdot 2^3+\cdots+(n-2)\cdot 2^{n-2}+(n-1)\cdot 2^{n-1}\right)-\tfrac{1}{4}\left(0\cdot 2^0+1\cdot 2^1\right)\\
&\quad= \tfrac{1}{4}\left((n-2)2^n+2\right)-\tfrac{1}{2}\qquad\text{(using the formula given)}\\
&\quad= (n-2)\cdot 2^{n-2}+\tfrac{1}{2}-\tfrac{1}{2}\\
&\quad= (n-2)\cdot 2^{n-2}
\end{aligned}$$

Therefore, the number of arrangements is $(n-2)\cdot 2^{n-2}$.

(c) Three solutions are included below. The first directly uses the result of part (b), the second is a direct count similar to the solution given to part (b), and the third is also a direct count, but it is more easily generalized.

Solution 1

Consider an arrangement $a_1, a_2, \ldots, a_n$ of the integers 1 through n, and fix a positive integer $1 \le m \le n$. For each i from 1 through n, define $b_i = a_i$ if $b_i < m$, and $b_i = a_i + 1$ if $b_i \ge m$. In other words, the list $b_1, b_2, \ldots, b_n$ is obtained by increasing the a_i that are at least m by 1, and leaving all other a_i unchanged.

The list $b_1, b_2, \ldots, b_n$ is an arrangement of $1, 2, \ldots, m-1, m+1, \ldots, n, n+1$. For example, the arrangement $4, 3, 5, 1, 6, 2$ of $1, 2, 3, 4, 5, 6$ would yield $5, 4, 6, 1, 7, 2$ when $m = 3$.

Suppose $a_1, a_2, \ldots, a_n$ has an internal peak at position k. Then $a_{k-1} < a_k$, so one of

- $b_{k-1} = a_{k-1}$ and $b_k = a_k$,
- $b_{k-1} = a_{k-1}$ and $b_k = a_k + 1$, or
- $b_{k-1} = a_{k-1} + 1$ and $b_k = a_k + 1$

must be true. Importantly, it is impossible for $b_{k-1} = a_{k-1} + 1$ and $b_k = a_k$ since if $a_{k-1} > m$, then $a_k > m$. In all three of the situations above, we get that $b_{k-1} < b_k$. Since the arrangement has an internal peak at position k, $a_{k+1} < a_k$ as well, and a very similar argument shows that $b_{k+1} < b_k$. Thus, the arrangement $b_1, b_2, \ldots, b_n$ has an internal peak at position k.

By very similar reasoning, one can show that if $b_1, b_2, \ldots, b_n$ has an internal peak at position k, then so must the arrangement $a_1, a_2, \ldots, a_n$.

Using part b, for every m with $1 \le m \le n$, there are $(n-2)\cdot 2^{n-2}$ arrangements of $1, 2, 3, \ldots, m-1, m+1, \ldots, n, n+1$ with an internal peak at position 2 and no other internal peaks.

We now turn our attention to counting the number of arrangements of $1, 2, 3, \ldots, n$ that have an internal peak at position 3 and no other internal peaks.

Consider an arrangement $a_1, a_2, \ldots, a_n$ with an internal peak at position 3 and no other internal peaks, and suppose $a_1 = m$. Then $a_2, a_3, \ldots, a_n$ is an arrangement of the integers $1, 2, 3, \ldots, m-1, m+1, \ldots, n$ with an internal peak at position 2 and no other internal peaks. From the earlier argument, there are exactly $((n-1)-2)\cdot 2^{(n-1)-2}$ such arrangements. We will now argue that each such arrangement arises in this way.

Consider an arbitrary arrangement $b_1, b_2, \ldots, b_{n-1}$ of $1, 2, 3, \ldots, m-1, m+1, \ldots, n$ with an internal peak at position 2 and no other internal peaks. The arrangement $m, b_1, b_2, \ldots, b_{n-1}$ of $1, 2, 3, \ldots, n$, has an internal peak at position 3 and has no internal peaks to the right of b_2. The only other possibility for an internal peak would be at position 2, but $b_2 > b_1$, so there is no internal peak at this position, regardless of the value of m.

Therefore, we can count the number of arrangements of $1, 2, 3, \ldots, n$ with an internal peak at position 3 and no other internal peaks as follows: Choose m to be any of $1, 2, 3, \ldots, n$ in n possible ways. Then choose an arrangement $b_1, b_2, \ldots, b_{n-1}$ of $1, 2, 3, \ldots, m-1, m+1, \ldots, n$ to have an internal peak at position 2 and no other internal peaks. This can be done in $(n-3)2^{n-3}$ ways. The arrangement $m, b_1, b_2, \ldots, b_{n-1}$ is an arrangement of $1, 2, 3, \ldots, n$ with an internal peak at position 3 and no other internal peaks. By the reasoning above, every such arrangement arises in this way, so there are $n(n-3)2^{n-3}$ arrangements.

Solution 2

Suppose that an arrangement of $1, 2, \ldots, n-1, n$ has an internal peak at position 3 and no other internal peaks.

Suppose further that the arrangement begins a, b, c, d.

Since position 3 is a peak, then $b < c$ and $c > d$.

Note that, as in (b), this means that $3 \leq c \leq n$.

Since there is no peak to the right of position 3, then the $n-3$ numbers to the right of c are either strictly decreasing from d, strictly increasing from d, or decreasing then increasing with no other change of direction.

We need to consider two possible cases: $a < c$ or $a > c$.

Case 1: $a < c$

Here, $c \neq 3$, since $a < c$ and $b < c$ and $d < c$. Thus, suppose that $4 \leq c \leq n$.

There are $c-1$ choices for a (the only restriction is that it is less than c).

There are then $c-2$ choices for b (less than c and cannot equal a).

The numbers $c+1, c+2, \ldots, n$ need to be placed in increasing order at the rightmost end of the arrangement, as we saw in similar situations in (b).

The remaining $c-3$ numbers (all those less than c but not a or b) need to be arranged between position 4 and position c with the directional requirements above. There are 2^{c-4} ways to do this.

Therefore, in this case, there are $\displaystyle\sum_{c=4}^{n}(c-1)(c-2)2^{c-4}$ arrangements.

Re-indexing this sum by letting $k = c-4$, the number of arrangements is $\displaystyle\sum_{k=0}^{n-4}(k+3)(k+2)2^{k}$.

Case 2: $a > c$

Here, $c \neq n$ since $a > c$. Thus, suppose that $3 \leq c \leq n-1$.

There are $n-c$ choices for a (the only restriction is that it is greater than c).

There are $c-1$ choices for b (less than c).

The numbers $c+1, c+2, \ldots, n$ (except for a) need to be placed in increasing order in the positions at the rightmost end of the arrangement, as in (b).

The remaining $c-2$ numbers (all those less than c but not b) need to be arranged between position 4 and position $c+1$ with the directional requirements above. There are 2^{c-3} ways to do this.

Therefore, in this case, there are $\displaystyle\sum_{c=3}^{n-1}(n-c)(c-1)2^{c-3}$ arrangements.

Letting $k = c-3$, the number of arrangements is $\displaystyle\sum_{k=0}^{n-4}(n-k-3)(k+2)2^{k}$.

Adding the sub-totals from the two cases, the total number of arrangements is

$$\begin{aligned}
&\sum_{k=0}^{n-4}(k+3)(k+2)2^k + \sum_{k=0}^{n-4}(n-k-3)(k+2)2^k \\
&\qquad = \sum_{k=0}^{n-4}(k+2)2^k(k+3+n-k-3) \\
&\qquad = \sum_{k=0}^{n-4} n(k+2)2^k \\
&\qquad = n\sum_{k=0}^{n-4} k2^k + n\sum_{k=0}^{n-4} 2^{k+1} \\
&\qquad = n((n-5)2^{n-3}+2) + n(2^{n-2}-2) \qquad \text{(given formula and geometric series)} \\
&\qquad = n(n-5)2^{n-3} + 2n2^{n-3} \\
&\qquad = n(n-3)2^{n-3}
\end{aligned}$$

Therefore, the total number of arrangements with an internal peak at position 3 and no other internal peaks is $n(n-3)2^{n-3}$. (When $n = 5$, we obtain $5 \cdot 2 \cdot 2^2 = 40$, as in (a).)

Solution 3

We want to count the number of arrangements of $1, 2, \ldots, n-1, n$ with an internal peak at position 3 and no other internal peaks.

We start by arranging the numbers $1, 2, \ldots, n-1$; we will insert n later.

We note that, if n is in the interior of an arrangement, it will be a peak. Therefore, for an arrangement to have a peak at position 3 and no other internal peaks, then either n is in position 3 or it is at one of the ends of the arrangement.

Choose 2 numbers from the list $1, 2, \ldots, n-1$ to be the first 2 numbers of an arrangement of $n-1$ numbers. There are 2 orders in which we can arrange these 2 numbers. We call the first number in the arrangement a and the second number b. In principle, we can have either $a < b$ or $a > b$ and still have a peak at position 3.

Let T be an arrangement of the $n-3$ numbers that remain from the list $1, 2, \ldots, n-1$. As we saw in (b), there are 2^{n-4} arrangements T as a list that is either strictly decreasing, or strictly increasing, or decreasing then increasing. (As in (b), we start from the largest number and place it on one end of the arrangement, and proceed through the remaining numbers in descending order.)

Once the 2 initial numbers are chosen, there are thus $2 \cdot 2^{n-4} = 2^{n-3}$ ways of arranging the 2 initial elements followed by the next $n-3$ elements into one arrangement of $n-1$ numbers. We now want to consider how to turn these arrangements of $n-1$ numbers into arrangements of n numbers that have a peak in position 3.

Let y be the leftmost number of the sublist T, which is currently in position 3.

Case 1: $b < y$

Since T is either strictly decreasing, strictly increasing, or decreasing then increasing, the arrangement of $n-1$ numbers already has a peak at position 3 (because $b < y$) unless the numbers in T are strictly increasing.

Here, we can insert the integer n into this arrangement of $n-1$ numbers to create or maintain a peak at position 3 in the following ways:

- At position 3 between b and y. This ensures a peak at position 3 (regardless of whether the numbers in T are strictly increasing) and does not create another peak because of the shape of the arrangement on either side of position 3.

- At position n as long as the numbers in T are not strictly increasing. This ensures a peak at position 3. If $b < y$, the numbers in T are strictly increasing, and n is at position n, there will not be a peak at position 3.

There is no other position where n can be inserted to maintain an internal peak only at position 3. If n is inserted at the beginning of the arrangement, the possible peak moves to position 4; if n is inserted elsewhere in the middle of the arrangement, it will create a peak itself.

Case 2: $b > y$

Here, the arrangement of $n-1$ numbers already has a peak at position 2 unless $a > b$. Here, we can insert the integer n into this arrangement of $n-1$ numbers to create or maintain a peak at position 3 in the following ways:

- At position 3 between b and y. This ensures a peak at position 3 (regardless of whether $a < b$ or $a > b$) and does not create another peak because of the shape of the arrangement on either side of position 3.
- At position 1 as long as $a < b$. This moves the peak previously at position 2 to position 3. If $b > y$, n is at position 1 and $a > b$, there will not be a peak at position 3.

Therefore, for each of the $\binom{n-1}{2}$ ways of choosing the first 2 numbers and the subsequent 2^{n-3} ways of arranging the initial $n-1$ numbers, there are 2 ways of inserting n, noting that there are two exceptional cases that we still need to remove. This gives $2 \cdot 2^{n-3} \cdot \binom{n-1}{2}$ or $2^{n-2}\binom{n-1}{2}$ arrangements, with two cases to sort out.

The arrangements that have been counted above that should not be counted in the final total are:

- Those with $b < y$, the arrangement of numbers in T strictly increasing, and n placed in position n. The direction of this arrangement of n numbers is either strictly increasing from position 1 to n, or decreasing from 1 to 2 and then increasing from 2 to n.
- Those with $b > y$, $a > b$, and n placed in position 1. The direction of this arrangement of n numbers is either strictly decreasing from position 1 to n, or decreasing from 1 to some position at or after position 3 and then increasing to position n.

These arrangements are exactly the arrangements of the integers $1, 2, \ldots, n-1$ that are strictly increasing (included in first bullet), strictly decreasing (included in second bullet), or decreasing then increasing with no other change of direction, with the integer n put at either the beginning or the end, as appropriate. With $n-1$ numbers, there are exactly 2^{n-2} such arrangements.

This means that the total number of arrangements with an internal peak at position 3 and no other internal peaks is $2^{n-2}\binom{n-1}{2} - 2^{n-2}$ or $2^{n-2}\left(\binom{n-1}{2} - 1\right)$.

This argument can be generalized to determine the number of arrangements with exactly one internal peak at position k.