The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2026
Canadian Team Mathematics Contest

April 2026

Solutions

Individual Problems

1. We have that $3^2 + 2^3 = 9 + 8 = 17$.

ANSWER: 17

2. The leading digit can either be 6 or 2.
If the leading digit is 6, then there are three ways to arrange the remaining digits $0, 2, 2$.
If the leading digit is 2, then there are six ways to arrange the remaining digits $0, 2, 6$.
Therefore there are $6 + 3 = 9$ numbers.

ANSWER: 9

3. One Earth-day has $24 \times 60 = 1440$ Earth-minutes. Since each Cemtece-hour (or C-hour) has 15-Earth minutes, one Earth-day has $\frac{1440}{15} = 96$ C-hours. Since each C-day has 16 C-hours, we conclude that one Earth-day has $\frac{96}{16} = 6$ C-days.

ANSWER: 6

4. Since $a^4 = 3$, we have that $a^2 = \pm\sqrt{3}$. We assume that $a^2 = \sqrt{3}$ - it can be checked that you get the same answer we are about to get if you instead assumed $a^2 = -\sqrt{3}$. We have that

$$\left(a^2 + \frac{1}{a^2}\right)^2 = \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)^2 = 3 + 2 + \frac{1}{3} = \frac{16}{3}$$

ANSWER: $\frac{16}{3}$

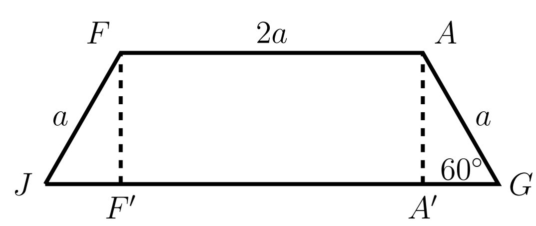
5. We want to determine how many positive integer values of d there are such that 900 has a factor of d^2. This is equivalent to 30 having a factor of d. The number 30 has eight positive divisors: 1, 2, 3, 5, 6, 10, 15, and 30. Therefore 900 has eight positive divisors that are perfect squares (they are the squares of the 8 numbers listed.)

ANSWER: 8

6. We have that $x = 11z - 2y = 11(2x - 3y) - 2y = 22x - 35y$. This rearranges to give $35y = 21x$, so $5y = 3x$, so $\dfrac{x}{y} = \dfrac{5}{3}$. (Note we use the fact that $y \neq 0$ in the final step.)

ANSWER: $\frac{5}{3}$

7. The hexagon being regular means that all six side-lengths are equal, and all six interior angles are equal. Since trapezoids $FAGJ$ and $BCHG$ are congruent, we must have $AG = GB$, that is, G is the midpoint of AB, and $FA = 2AG$. Similar identities hold around the rest of the hexagon.
Note that $\triangle GHJ$ is equilateral, with each interior angle measuring $60°$. Therefore $\angle AGJ = \frac{1}{2}(180° - 60°) = 60°$. Let $AG = FJ = a$ and hence $AF = 2a$. Consider the following diagram where A' is on JG such that AA' is perpendicular to JG, and similarly for F'.



We have that $GA' = JF' = a\cos(60°) = \frac{a}{2}$. Since $JG = 3$ we therefore have that $\frac{a}{2}+2a+\frac{a}{2} = 3$, so $a = 1$. That is, the side length of the hexagon is 2.
A regular hexagon with side length 2 can be decomposed into six congruent, equilateral triangles, with side length 2. Thus the area of the hexagon is six times the area of an equilateral triangle with side length 2. By the Pythagorean Theorem, if we take the base as 2, then the height is $\sqrt{2^2-1} = \sqrt{3}$. So the area of one of the equilateral triangles is $\dfrac{2(\sqrt{3})}{2} = \sqrt{3}$. Thus the area is of our hexagon (made up of six equilateral triangles) is $6\sqrt{3}$.

Answer: $6\sqrt{3}$

8. The x-coordinates of A and B are the roots of $2x^2+(k-16)x+(14-k) = 0$, which are the same as the roots of $x^2 + \left(\frac{k-16}{2}\right)x + \left(\frac{14-k}{2}\right) = 0$. A quadratic equation $x^2+ax+b = 0$ has the property that the sum of its roots equals $-a$. Let r, s denote the roots $x^2 + \left(\frac{k-16}{2}\right)x + \left(\frac{14-k}{2}\right) = 0$. Therefore $r+s = \frac{16-k}{2}$. But we also have that $\frac{r+s}{2} = \frac{7}{2}$ from the given information. So $r+s = 7$, so $14 = 16-k$, so $k = 2$.

We now solve the quadratic: with $k = 2$ the quadratic is $x^2-7x+6 = 0$, which has roots 1 and 6. When $x = 1$, the y-coordinate is $2(1)+14 = 16$, so $A = (1, 16)$. When $x = 6$, the y-coordinate is $2(6)+14 = 26$, so $B = (6, 26)$.

The distance between A and B is $\sqrt{(6-1)^2+(26-16)^2} = \sqrt{125} = 5\sqrt{5}$.

Answer: $5\sqrt{5}$

9. The largest possible number of Bs arises when the first B occurs as early as possible, and there are as few As as possible: one such word is $BAABAABAABAABAABAA$, which has 6 Bs.
The smallest possible number of Bs arises when the first B occurs as late as possible, and there are as many As as possible: one such word is $AAABAAABAAABAAABAA$, which has 4 Bs.
Therefore, any such word either has 4, 5, or 6 Bs. We consider these cases separately.

First, suppose there are 6 Bs. Between every two Bs there are at least two As, and thus every such word can be built from $BAABAABAABAABAAB$ by inserting 2 additional As. For this case, and the others, we say that an A is 'inside' if there is a B both to the left and to the right of it, neither required to be immediately to the left or to the right, and 'outside' otherwise.
If both of the additional As are placed on the outside, this may be done in 3 ways: $AAB\cdots B$, $AB\cdots BA$, or $B\cdots BAA$.
If one of the additional As is on the outside, there are 2 ways to place it, and 5 ways to place the inside A (it can go into any of the 5 'blocks' of As already present), and thus there are $2\times 5 = 10$ words in this case
If both of the additional As are put inside, this can be done in $\binom{5}{2} = 10$ ways, since two As cannot go into the same block (as there would otherwise be 4 consecutive As).
In summary, when there are 6 Bs, there are $3+10+10 = 23$ words contributing to our count.

Now suppose there are 5 Bs. Every such word can be built from $BAABAABAABAAB$ by inserting 5 additional As.
Since we can't have more than 3 consecutive As (we won't reference this constraint again, though we will use it many more times), there are 2 ways to place these 5 additional As all on the outside: namely $AAAB\cdots BAA$ or $AAB\cdots BAAA$.

Suppose there are 4 outside As and 1 inside A. There are 3 ways to place 4 outside As, and 4 ways to place 1 inside A, thus there are $3 \times 4 = 12$ words in this case.
Suppose there are 3 outside As and 2 inside As. There are 4 ways to place the outside As, and $\binom{4}{2} = 6$ ways to place the inside As, for a total of 24 words in this case.
Suppose there are 2 outside As and 3 inside As. There are 3 ways to place the outside As, and $\binom{4}{3} = 4$ ways to place the inside As, for a total of 12 words in this case.
Suppose there is 1 outside A and 4 inside As. There are 2 ways to place the outside A, and only 1 way to place the inside As (namely each block gains an A), and thus there are 2 words in this case.
There cannot be 5 inside As since this would violate the constraint.
Thus, when there are 5 Bs, the total number of words is $2 + 12 + 24 + 12 + 2 = 52$.

Finally, assume there are 4 Bs. Every such word can be built from $BAABAABAAB$ by inserting 8 additional As. Since no more than 6 As may be placed on the outside, and no more than 3 As may be placed on the inside, there are two cases to consider: 6 As outside and 2 As inside, or 5 As outside and 3 As inside.
There is 1 way to place 6 As outside, and $\binom{3}{2} = 3$ ways to place 2 As inside, for a total of 3 words in this case.
There are 2 ways to place 5 As outside, and $\binom{3}{3} = 1$ way to place 3 As inside, for a total of 2 words in this case.
Thus, when there are 4 Bs, there are $3 + 2 = 5$ words in this case.

In summary, the total number of words are $23 + 52 + 5 = 80$.

ANSWER: 80

10. We will use the following fact: suppose we are given two (infinite in both directions) arithmetic sequences with common differences d_1 and d_2, let $L = \text{lcm}(d_1, d_2)$. Suppose c is a common term in both sequences. Then the common terms of both sequences are precisely those terms of the form $c + nL$ with $n \in \mathbb{Z}$. In particular, the common term preceding c is $c - L$, and the common term preceding $c - L$ is $c - 2L$.

Since 233 and 2258 are both terms of $b_1, b_2, b_3, \ldots,$ we must have that d is a divisor of $2258 - 233 = 2025 = 45^2 = 3^4 5^2$. We may assume $d > 0$, since replacing d with $-d$ doesn't change which terms occur in the sequence, and thus the possible values of d are precisely the positive divisors of 2025. The positive divisors of 2025 are precisely those numbers of the form $3^a 5^b$ where $0 \le a \le 4$ and $0 \le b \le 2$.

We require that the second smallest term, larger than 2026, which appears in both sequences is 2258. Let $L = \text{lcm}(45, d)$. By the discussion in the first paragraph, the desired condition is equivalent to $2258 - L > 2026$ and $2258 - 2L \le 2026$. These are equivalent to $116 \le L \le 232$. We now simply check whether or not this condition is satisfied among the divisors of 2025. This is summarized in the following table:

d (divisor of 2025)	$L = \text{lcm}(45, d)$	Is $116 \leq L < 232$?
$3^0 5^0 = 1$	45	✗
$3^1 5^0 = 3$	45	✗
$3^2 5^0 = 9$	45	✗
$3^3 5^0 = 27$	135	✓
$3^4 5^0 = 81$	405	✗
$3^0 5^1 = 5$	45	✗
$3^1 5^1 = 15$	45	✗
$3^2 5^1 = 45$	45	✗
$3^3 5^1 = 135$	135	✓
$3^4 5^1 = 405$	405	✗
$3^0 5^2 = 25$	225	✓
$3^1 5^2 = 75$	225	✓
$3^2 5^2 = 225$	225	✓
$3^3 5^2 = 675$	675	✗
$3^4 5^2 = 2025$	2025	✗

Therefore, the sum of all possible d values is $27 + 135 + 25 + 75 + 225 = 487$

Answer: 487

Team Problems

1. The front face of the prism has area $(1 + 2) \times (1 + 1 + 1) = 9$. The depth of the prism is $1 + 2 + 1 = 4$, so the total volume of the prism is $9 \times 4 = 36$.

   The volume of each brick is $1 \times 1 \times 2 = 2$, so there are $\frac{36}{2} = 18$ bricks forming the prism.

   Answer: 18

2. The fraction $\frac{100}{a}$ is largest when its denominator, a, is as small as possible. Thus, the maximum value of $\frac{100}{a}$ with $4 \le a \le 20$ is $\frac{100}{4} = 25$.

   Answer: 25

3. A circle with radius 2 has area $\pi(2)^2 = 4\pi$. The side length of a square with area 4π is $\sqrt{4\pi} = 2\sqrt{\pi}$.

   Answer: $2\sqrt{\pi}$

4. From the diagram, the angles y°, $2x^\circ$, x°, and $3x^\circ$, together with the given right angle, have a sum of 360°. Thus, $y + 2x + x + 3x + 90 = 360$ which is equivalent to $y + 6x = 270$. It is given that $x + y = 60$, so $x = 60 - y$. Substituting into $y + 6x = 270$ gives $y + 6(60 - y) = 270$ or $y + 360 - 6y = 270$. Solving for y gives $y = 18$.

   Answer: 18

5. Suppose the hundreds digit is A and the units digit is B. Then $N = 100A + B$ (since the tens digit is given to be 0). The integer obtained by reversing the digits is $100B + A$, which is 495 greater than N. Therefore, we have

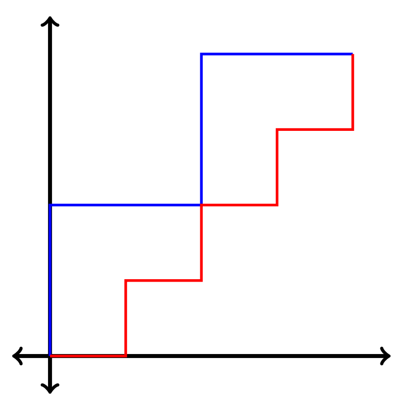
   $$100A + B + 495 = A + 100B$$

   which can be rearranged to get $495 = 99B - 99A = 99(B - A)$. Dividing both sides by 99 gives $5 = B - A$.

   It is given in the question that $A + B = 11$, which we can add to $A - B = 5$ to get $2B = 16$ or $B = 8$, from which it follows that $A = 3$. Therefore, $N = 308$.

   Answer: 308

6. The first four blue line segments and the first eight line segments are drawn below.



The blue and red lines will first meet at $(2, 2)$, then at $(4, 4)$, then at $(6, 6)$, and so on so that they meet at every point (n, n) where n is an even positive integer.

The area enclosed by the two lines from (n, n) to $(n+2, n+2)$ is a square of side length 2 with a square of side length 1 removed, and so has area $2^2 - 1^2 = 3$.

There are 2026 such regions in all, so the total enclosed area is $3 \times 2026 = 6078$.

Answer: 6078

7. We write $\frac{65}{49} = 1 + \frac{16}{49}$, and therefore we have that $1 + \frac{16}{49} = 1 + \frac{1}{a+\frac{1}{b}}$. Therefore, $\frac{16}{49} = \frac{1}{a+\frac{1}{b}}$. Taking reciprocals, we get $\frac{49}{16} = a + \frac{1}{b}$, or $3 + \frac{1}{16} = a + \frac{1}{b}$. Therefore $a = 3$ and $b = 16$ work. We now show that these are the only values of a and b which work.

   Since b is a positive integer, we have $0 < \frac{1}{b} \leq 1$, so $a < a + \frac{1}{b} \leq a + 1$. Since $a + \frac{1}{b} = 3 + \frac{1}{b}$, we thus have that $a < 3 + \dfrac{1}{b} \leq a + 1$. Since a is the only positive integer that satisfies this, we must have that $a = 3$. It then follows that we must have $b = 16$.

   Answer: $(a, b) = (3, 16)$

8. Circle D has radius 3 and circle C has radius 2, so their circumferences are 6π and 4π, respectively. This means 2 times the circumference of circle D is 12π, which equals 3 times the circumference of circle C. Thus, when circle D completes exactly two revolutions, circle C will have completed exactly 3 revolutions. Therefore, the rate at which circle C turns is exactly $\dfrac{3}{2}$ times the rate at which circle D turns.

   By similar reasoning, since the radii of B and C are 5 and 2 respectively, the rate at which circle B turns is $\dfrac{2}{5}$ times the rate at which circle C turns. Also, since the radius of circle A is 1 and the radius of circle B is 5, circle A turns at $\dfrac{5}{1} = 5$ the rate at which circle B turns.

   Putting this together and using that circle D turns at 1 revolution per second, we have that circle C turns at $\dfrac{3}{2}$ revolutions per second, circle B turns at $\dfrac{2}{5} \times \dfrac{3}{2} = \dfrac{3}{5}$ revolutions per second, and circle A turns at $5 \times \dfrac{3}{5} = 3$ revolutions per second.

   Answer: 3 (or -3)

9. Rearranging the equation $2a + 5b + 10c = 155$, we get $2a = 155 - 5b - 10c = 5(31 - b - 2c)$, and so $2a$ is a multiple of 5. Since 5 is prime and 2 is not a multiple of 5, we must have that a is a multiple of 5. The only prime number that is a multiple of 5 is 5 itself, so we conclude that $a = 5$.

   Now that we have $a = 5$, we can substitute into $2a+5b+10c = 155$ to get $10+5b+10c = 155$, from which it follows that $b+2c = 29$. Adding this to the equation $c-b = 4$ gives $3c = 29+4 = 33$, so $c = 11$. From $c - b = 4$, we get $b = 11 - 4 = 7$.

   Answer: $(a, b, c) = (5, 7, 11)$

10. Such a sequence must go right 7 times and down 2 times, and is determined by the ordering of these. Such a sequence can be constructed by imagining 9 blank spaces in a row, putting D into two of them (and then putting R into the remaining seven of them). However, we are not allowed the sequence $DDRRRRRRR$, since this would mean going under A. Therefore our final answer is $\binom{9}{2} - 1 = 36 - 1 = 35$.

ANSWER: 35

11. We show the effects of running the `swap`s a small number of times (the output of one row is the input of the next row):

$$\texttt{CANDY} \xrightarrow{\texttt{swap(1,2)}} \texttt{ACNDY} \xrightarrow{\texttt{swap(3,5)}} \texttt{ACYDN} \xrightarrow{\texttt{swap(2,4)}} \texttt{ADYCN}$$
$$\texttt{ADYCN} \xrightarrow{\texttt{swap(1,2)}} \texttt{DAYCN} \xrightarrow{\texttt{swap(3,5)}} \texttt{DANCY} \xrightarrow{\texttt{swap(2,4)}} \texttt{DCNAY}$$
$$\texttt{DCNAY} \xrightarrow{\texttt{swap(1,2)}} \texttt{CDNAY} \xrightarrow{\texttt{swap(3,5)}} \texttt{CDYAN} \xrightarrow{\texttt{swap(2,4)}} \texttt{CAYDN}$$
$$\texttt{CAYDN} \xrightarrow{\texttt{swap(1,2)}} \texttt{ACYDN} \xrightarrow{\texttt{swap(3,5)}} \texttt{ACNDY} \xrightarrow{\texttt{swap(2,4)}} \texttt{ADNCY}$$
$$\texttt{ADNCY} \xrightarrow{\texttt{swap(1,2)}} \texttt{DANCY} \xrightarrow{\texttt{swap(3,5)}} \texttt{DAYCN} \xrightarrow{\texttt{swap(2,4)}} \texttt{DCYAN}$$
$$\texttt{DCYAN} \xrightarrow{\texttt{swap(1,2)}} \texttt{CDYAN} \xrightarrow{\texttt{swap(3,5)}} \texttt{CDNAY} \xrightarrow{\texttt{swap(2,4)}} \texttt{CANDY}$$

That is, when each of the three `swap` commands is run 6 times (in the specified order), the effect is the same as doing nothing. (And 6 is the smallest number with this property.)
We have that $2026 = 6(337) + 4$. Therefore the effect of repeating 2026 times is the same as the effect of repeating 4 times. From our work above, this gives us a final answer of `ADNCY`.

ANSWER: `ADNCY`

12. We must have that c has a factor of 11 for such an integer c. c cannot have only one digit. If c has two digits, then both digits of c must be equal since c has a factor of 11. Then, for some positive digit A, $c = 10A + A = 11A$. If we had $c = 11 \times S_c$, we would then have $11A = 11(2A)$, which is a contradiction. Thus c must have at least three digits.
We look for c having three digits. Write $c = ABC$ with A, B, and C the respective digits of c. The condition $c = 11 \times S_c$ says that $100A + 10B + C = 11(A + B + C)$, which is equivalent to $89A = B + 10C$. If $A = 1$, this gives $C = 8$ and $B = 9$, which gives $c = 198$. Since $11 \times S_c = 11 \times (1 + 9 + 8) = 198$, and we are told that there is only one such c, we conclude that 198 is our final answer.

Remark: There is a faster solution for a student who knows some congruence theory. The rule for divisibility by 9 says that $c \equiv S_c \mod 9$ (for all positive integers c). Thus we are told that $11c \equiv c \mod 9$, from which we conclude that $c \equiv 0 \mod 9$. Since $c \equiv 0 \mod 11$ also, we conclude that $c \equiv 0 \mod 99$. After checking that 99 doesn't work, we find that 198 does work. [It is also not too hard to show that there is only one such c.]

ANSWER: 198

13. Let A, B, and D have coordinates $(0, 0)$, $(12, 0)$ and $(0, 12)$ respectively. Then E has coordinates $(e, 0)$ for some $0 < e < 6$, F has coordinates $(e + 6, 0)$, and G has coordinates $(e + 6, 12)$.
Note line segment DB has equation $y = 12-x$, and line segment EG has equation $y = 2(x-e)$.
Point M is determined by intersecting the two lines describe above. Doing this gives us that $M = (4 + \frac{2}{3}e, 8 - \frac{2}{3}e)$.
Point N is determined by the intersection of $y = 12 - x$ with the vertical line $x = e + 6$. This gives us that $N = (e + 6, 6 - e)$.
The area of $\triangle EFG$ is $\dfrac{6(12)}{2} = 36$. Therefore the given condition is equivalent to saying that

$\triangle GNM$ has area 18.
We view NG has the base of $\triangle GNM$. The length of this base is $12 - (6 - e) = 6 + e$. The corresponding height is the (horizontal) distance from M to the vertical line $x = e + 6$, which is $(e + 6) - (4 + \frac{2}{3}e) = 2 + \frac{1}{3}e$.
Therefore we have that $18 = \dfrac{(6 + e)(2 + \frac{1}{3}e)}{2}$. This is equivalent to $e^2 + 12e - 72 = 0$. Using the quadratic formula, we find that the only positive solution to this is $e = 6(\sqrt{3} - 1)$. Therefore our final answer is

$$\frac{AB}{AE} = \frac{12}{6(\sqrt{3} - 1)} = \sqrt{3} + 1$$

Answer: $\sqrt{3} + 1$ (or $\frac{2}{\sqrt{3}-1}$)

14. Suppose the common remainder is r. Then there are integers a, b, and c such that

$$332 = ad + r$$
$$456 = bd + r$$
$$549 = cd + r$$

Subtracting the first equation from the second gives $456-332 = bd+r-ad-r$ or $124 = d(b-a)$. Similarly, subtracting the first equation from the third gives $217 = d(c - a)$, and subtracting the second equation from the third gives $93 = d(c - b)$.

We have that $d(b - a) = 124$, $d(c - a) = 217$, and $d(c - b) = 93$, so d must be a divisor of 124, 217, and 93. The only positive divisors of 93 are 1, 3, 31, and 93. Of these, only 1 and 31 are divisors of 124, so since $d > 1$, we must have that $d = 31$. Note that $217 = 7 \times 31$, so 31 is indeed a divisor of 217 as well.

Answer: 31

15. Since Bottle A has 40 mL, 10% of which is acid, then Bottle A has 4 mL of acid. Similarly, Bottle B has 10 mL of acid and Bottle C has 15 mL of acid. That is, in total, there are $4 + 10 + 15 = 29$ mL of acid among the three bottles.
The mixture with volume 60 mL, 25% of which is acid, must have 15 mL of acid. Therefore there are $29 - 15 = 14$ mL of acid remaining among the remaining contents of the three bottles. The new mixture has volume $40 + 50 + 50 - 60 = 80$. Therefore the new mixture has an acid percentage of $100 \times \frac{14}{80}\% = 17.5\%$.

Answer: 17.5%

16. Suppose $t_1 = a$ and the common ratio is r so that $t_2 = ar$, $t_3 = ar^2$, and in general, $t_k = ar^{k-1}$ for $1 \le k \le n$.

The equation $t_1 t_n = 3$ gives $a^2 r^{n-1} = 3$. The condition that $t_1 t_2 \cdots t_n = 6561 = 3^8$ gives $a^n r^{1+2+3+4+\cdots+(n-1)} = a^n r^{(n^2-n)/2} = 3^8$.

Raising both sides of $a^2 r^{n-1} = 3$ to the power of n gives $a^{2n} r^{n^2-n} = 3^n$, and raising both sides of $a^n r^{(n^2-n)/2} = 3^8$ to the power of 2 gives $a^{2n} r^{n^2-n} = 3^{16}$. Hence, we have $a^{2n} r^{n^2-n} = 3^n$ and $a^{2n} r^{n^2-n} = 3^{16}$, from which it follows that $n = 16$.

Answer: $n = 16$

17. The inequality $4\sin x\cos x + 2\sin x > 2\sqrt{3}\cos x + \sqrt{3}$ may be rewritten as $4\sin x\cos x + 2\sin x - 2\sqrt{3}\cos x - \sqrt{3} > 0$, which may be rewritten as

$$(2\sin x - \sqrt{3})(2\cos x + 1) > 0$$

This inequality is true when both terms are positive or both terms are negative. In what follows, we only consider x values as integers satisfying $0 \le x < 360$. We have that

$$2\sin x - \sqrt{3} > 0 \iff \sin x > \frac{\sqrt{3}}{2} \iff x > 60 \text{ and } x < 120$$

$$2\sin x - \sqrt{3} < 0 \iff \sin x < \frac{\sqrt{3}}{2} \iff x < 60 \text{ or } x > 120$$

$$2\cos x + 1 > 0 \iff \cos x > -\frac{1}{2} \iff x < 120 \text{ or } x > 240$$

$$2\cos x + 1 < 0 \iff \cos x < -\frac{1}{2} \iff x > 120 \text{ and } x < 240$$

Therefore $2\sin x - \sqrt{3}$ and $2\cos x + 1$ are both positive precisely when $x > 60$ and $x < 120$ - these are the values $61, 62, \ldots, 119$, of which there are 59.
We also have that $2\sin x - \sqrt{3}$ and $2\cos x + 1$ are both negative precisely when $x > 120$ and $x < 240$ - these are the values $121, 122, \ldots, 239$, or which there are 119.
Therefore there are a total of $59+119 = 178$ values of x such that $(2\sin x - \sqrt{3})(2\cos x + 1) > 0$.
Since there are 360 possible values of x, the probability is $\dfrac{178}{360} = \dfrac{89}{180}$.

ANSWER: $\frac{89}{180}$

18. Note that

$$42xyz + 21yz + 14xz + 6xy + 2x + 3y + 7z + 1 = (2x+1)(3y+1)(7z+1)$$

Since we want this product to be as small as possible, we first consider x, y, and z to be 1, 2, and 3 in some order. Note that $z \neq 1$ since this N has a factor of 8, and hence has three factors of 2. Similarly, $y \neq 1$. Thus, among these 6 options, we only need to consider $(x, y, z) = (1, 2, 3)$ and $(x, y, z) = (1, 3, 2)$. For $(x, y, z) = (1, 2, 3)$ we have that $N = 3 \times 10 \times 15$ which has two factors of 3. For $(x, y, z) = (1, 3, 2)$, we have that $N = 3 \times 7 \times 22 = 2 \times 3 \times 7 \times 11 = 462$, which is a product of distinct primes.

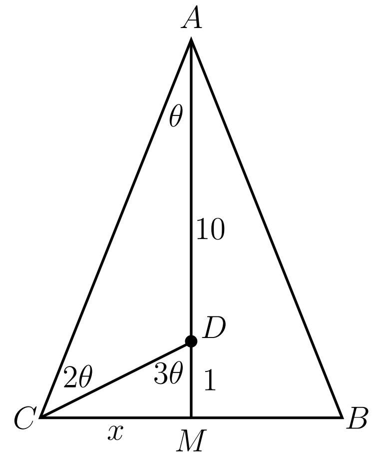
We now show this is minimal. If x, y, z are not $1, 2, 3$, in some order, then we must have that at least one of them is 4 or more. The minimal value of N here occurs with $x = 4$, $y = 2$, and $z = 1$, so

$$N = (2x+1)(3y+1)(7z+1) \ge (2(4)+1)(3(2)+1)(7(1)+1) = 504$$

showing that 462 is indeed minimal.

ANSWER: 462

19. Since the triangle is isosceles at A and M is the midpoint of BC, we also have that AM bisects $\angle BAC$ and DM bisects $\angle BDC$. Since $\angle BDC = 3\angle BAC$, we then also have that $\angle MDC = 3\angle MAC$. Let $\theta = \angle MAC$. Then $\angle CDA = 180 - 3\theta$, and therefore $\angle DCA = 2\theta$. Let $x = CM$. Here is our diagram, not drawn to scale:



The law of sines on $\triangle ADC$ tells us that $\dfrac{\sin\theta}{\sqrt{1+x^2}} = \dfrac{\sin(2\theta)}{10}$. Since $\sin(2\theta) = 2\sin\theta\cos\theta$ (and $\sin\theta \neq 0$), we therefore have that $\cos\theta = \dfrac{5}{\sqrt{1+x^2}}$. However, from $\triangle AMC$, we also have that $\cos\theta = \dfrac{11}{\sqrt{11^2+x^2}}$. Therefore $\dfrac{5}{\sqrt{1+x^2}} = \dfrac{11}{\sqrt{11^2+x^2}}$.

Squaring both sides and then solving for x^2 yields $x^2 = \frac{121}{4}$ and, since $x > 0$, $x = \frac{11}{2}$. Since $\triangle AMC$ is right-angled with $AM = 11$, we have that $AC^2 = CM^2 + AM^2$, and so $AC = \dfrac{11\sqrt{5}}{2}$. Therefore the perimeter of $\triangle ABC$ is $2\left(\frac{11\sqrt{5}}{2}\right) + 2\left(\frac{11}{2}\right) = 11(1+\sqrt{5})$

Answer: $11(1+\sqrt{5})$

20. Let $a_n = \log_{2^n}(3)$ and let $b_n = \log_{3^{n+1}}(2)$. We then have (from the definition of logarithms) that $(2^n)^{a_n} = 3$ and $(3^{n+1})^{b_n} = 2$. Combining these, we have that

$$3 = 2^{na_n} = (3^{(n+1)b_n})^{na_n} = 3^{n(n+1)a_nb_n},$$

from which we conclude that $n(n+1)a_nb_n = 1$ for all natural numbers n. In other words, $a_nb_n = \dfrac{1}{n(n+1)}$ for all n.

We are told that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = \dfrac{7}{8}$. From our work above, this says that

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{n(n+1)} = \frac{7}{8}.$$

We now use the identity $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. This turns the corresponding sum into

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{7}{8}$$

Notice that the $\frac{1}{2}$s cancel out, the $\frac{1}{3}$s cancel out, all the way up to the $\frac{1}{n}$s cancelling out. Therefore the expression simplifies into $1 - \dfrac{1}{n+1} = \dfrac{7}{8}$, from which we conclude that $n = 7$.

Answer: 7

21. Consider a pair of paths, each 4 moves long, taken respectively by ants A and B where they end at the same point. This can be transformed into a path where ant A walks from $(1, 2)$ to $(6, 5)$ by following As initial path of 4 moves, and then following the reversal of Bs initial path. Conversely, every path from $(1, 2)$ to $(6, 5)$ can be decomposed into pair of paths, each 4 moves long, which finish at the same point. Thus the given problem is equivalent to the following problem: what is the probability that an ant starting at point $(1, 2)$ ends up at point $(6, 5)$ after 8 randomly selected moves of 1 unit up or 1 unit right, each with probability $\frac{1}{2}$?

The ant has a total of 2^8 possible paths. To end up at $(6, 5)$, starting from $(1, 2)$, we require 5 of the 8 moves to be 'right' (and the other 3 to be 'up'). Thus the number of paths which terminate at $(6, 5)$ is $\binom{8}{5} = \dfrac{8!}{3!5!} = \dfrac{6 \times 7 \times 8}{3!} = 56$.

Therefore our final answer is $\dfrac{56}{2^8} = \dfrac{7}{32}$.

Remark: In fact, for the ants to meet, they must have each taken 4 steps. Let A denote the ant initially located at $(1, 2)$ and B denote the ant initially located at $(6, 5)$. Initially, A begins on the line $x + y = 3$, and after moving k times, A will be on the line $x + y = 3 + k$, since every move increases x or y by exactly one and leaves the other alone. Similarly, after moving k times, B will be on the line $x + y = 11 - k$. In particular, for them both to be at the same point after the same number of moves, we must have $3 + k = 11 - k$, i.e. $k = 4$. That is, for the ants to be at the same point (simultaneously), they must have each moved exactly 4 times.

ANSWER: $\frac{7}{32}$

22. Suppose we have n consecutive integers which sum to 475. Call the first integer a. Then we have $a + (a + 1) + \cdots + (a + (n - 1)) = 475$, which can be written as $na + \frac{n(n-1)}{2} = 475$, so $2an + n(n - 1) = 950$. Factoring out the n, the desired condition is equivalent to

$$n(2a + n - 1) = 950.$$

In particular, n must be a divisor of 950. We have that $950 = 2^1 5^2 19^1$, and so every divisor of 950 must be of the form $2^a 5^b 19^c$ with $0 \le a \le 1$, $0 \le b \le 2$, and $0 \le c \le 1$. That is, there are 12 divisors.

n (divisor of 950)	$2a + n - 1 (= 950/n)$	a
$2^0 5^0 19^0 = 1$	950	475
$2^0 5^0 19^1 = 19$	50	16
$2^0 5^1 19^0 = 5$	190	93
$2^0 5^1 19^1 = 95$	10	-42
$2^0 5^2 19^0 = 25$	38	7
$2^0 5^2 19^1 = 475$	2	-236
$2^1 5^0 19^0 = 2$	475	237
$2^1 5^0 19^1 = 38$	25	-6
$2^1 5^1 19^0 = 10$	95	43
$2^1 5^1 19^1 = 190$	5	-92
$2^1 5^2 19^0 = 50$	19	-15
$2^1 5^2 19^1 = 950$	1	-474

Since every value is an integer, our answer is the number of positive integer values of a, except for $a = 475$, since we required $n > 1$. We have 5 such values. For your interest, they are:

$$\begin{aligned} 16 + 17 + \cdots + 34 &= 475 \\ 93 + 94 + \cdots + 97 &= 475 \\ 7 + 8 + \cdots + 31 &= 475 \\ 237 + 238 &= 475 \\ 43 + 44 + \cdots + 52 &= 475 \end{aligned}$$

The negative values of a also give representations of 475 as a sum of consecutive integer, but without the first term being positive.

Answer: 5

23. Expanding the right sides of the two equations gives $x^2 + 2x + y^2 = k^2 - 1$ and $x^2 - 2x + y^2 = \left(\frac{1}{k}\right)^2 - 1$. These two equations can now be rearranged to get

$$\begin{aligned} (x+1)^2 + y^2 &= k^2 \\ (x-1)^2 + y^2 &= \left(\frac{1}{k}\right)^2 \end{aligned}$$

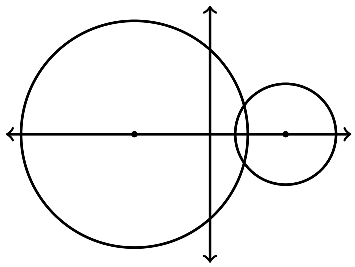
The first equation is that of a circle of radius k centred at $(-1, 0)$, and the second equation is that of a circle with radius $\frac{1}{k}$ centred at $(1, 0)$.

In order for these two circles to intersect at all, we must have that the sum of the radii is at least the distance between the centres, which is $\sqrt{(-1-1)^2 + (0-0)^2} = 2$. Observe that for positive k, we have

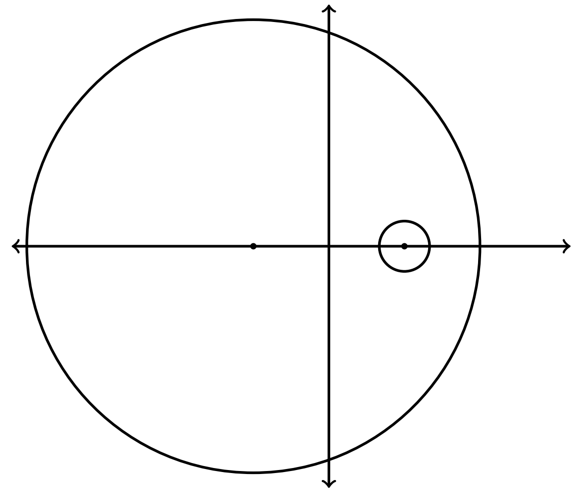
$$k + \frac{1}{k} = k + \frac{1}{k} - 2 + 2 = \left(\sqrt{k} - \frac{1}{\sqrt{k}}\right)^2 + 2 \geq 2$$

and so the sum of the radii is always at least 2. Note that $k + \frac{1}{k} = 2$ can be rearranged to $(k-1)^2 = 0$, so this minimum sum of radii is attained only when $k = 1$. When $k = 1$, the circles are tangent and will indeed intersect.

Let's consider what happens when $k > 1$. The diagram below and on the left shows the configuration of the circles when $k = 1.5$, and the diagram on the right shows the configuration of the circles when $k = 3$.



$k = 1.5$



$k = 3$

As $k > 1$ increases, the circle on the left grows and the circle on the right shrinks. They will intersect unless the circle of radius k (on the left) is so large that the circle of radius $\frac{1}{k}$ is contained completely within it. The rightmost point of the circle on the left has coordinates $(-1+k, 0)$, and the rightmost point of the circle on the right has coordinates $\left(1+\frac{1}{k}, 0\right)$.

Thus, when $k > 1$, the circles will intersect as long as $-1+k \leq 1+\frac{1}{k}$. Since $k > 1$, this inequality is equivalent to $-k+k^2 \leq k+1$ or $k^2-2k-1 \leq 0$. The coefficient of k^2 in the quadratic on the left of this inequality is positive, so the expression on the left represents a parabola that opens up. It is less than or equal to 0 when k is between the roots, which are $1-\sqrt{2}$ and $1+\sqrt{2}$.

Thus, under the assumption that $k > 1$, we get that $1-\sqrt{2} \leq k \leq 1+\sqrt{2}$ must hold in order for the circles to intersect. $1-\sqrt{2} \leq k$ is implied by the condition $k > 1$, so the only new piece of information is $k \leq 1+\sqrt{2}$. Thus, we get that all k satisfying $1 \leq k \leq 1+\sqrt{2}$ will cause the circles to intersect.

If $k < 1$, then we can consider what happens as k gets smaller and smaller. In this case, the circle with radius k (on the left) will shrink and the circle with radius $\frac{1}{k}$ will grow. This time comparing the leftmost points of the two circles, they will intersect as long as $-1-k \leq 1-\frac{1}{k}$ which is equivalent to $0 \leq k^2+2k-1$. The roots of k^2+2k-1 are $-1\pm\sqrt{2}$, and so $0 \leq k^2+2k-1$ when $k \leq -1-\sqrt{2}$ and when $k \geq -1+\sqrt{2}$. We are assuming that $k > 0$, so $k \leq -1-\sqrt{2}$ is not possible. We are also assuming that $k < 1$ in this case, so the k values of interest are those with $-1+\sqrt{2} \leq k < 1$.

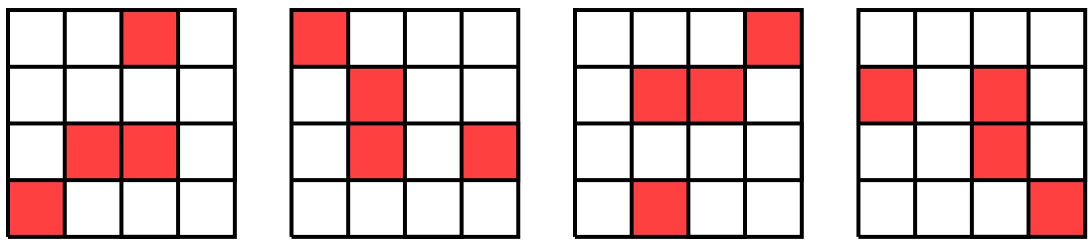
Combining this with the k values with $1 \leq k \leq 1+\sqrt{2}$, we get that the circles will intersect when k satisfies $-1+\sqrt{2} \leq k \leq 1+\sqrt{2}$.

ANSWER: $\left[-1+\sqrt{2}, 1+\sqrt{2}\right]$ (or $-1+\sqrt{2} \leq k \leq 1+\sqrt{2}$)

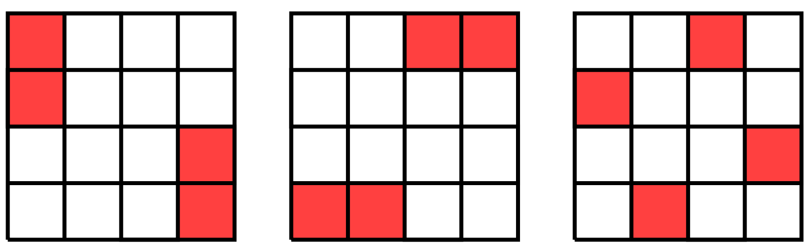
24. Not accounting for rotations, there are $\binom{16}{4} = \frac{16(15)14(13)}{24} = 2 \times 5 \times 14 \times 13 = 1820$ ways to colour four of the sixteen squares red. Every colouring falls into one of the following three groups:

- (Group 1) These are the colourings that have 90° (and 270°) rotational symmetry, meaning that it looks the same after rotated by 90°. Each colouring in Group 1 is equivalent to only itself.
- (Group 2) These are the colourings that have 180° (but not 90°) rotational symmetry. Each colouring in Group 2 is equivalent to exactly one other colouring (namely the colouring obtained by rotating by 90°).
- (Group 3) These are the colourings that are not in Group 1 or Group 2. This means you must rotate by 90° four times to obtain the original colouring. Each colouring in Group 3 is equivalent to exactly three other colourings.

For example, below are four equivalent colourings in Group 3 (and no other colouring is equivalent to any of them).

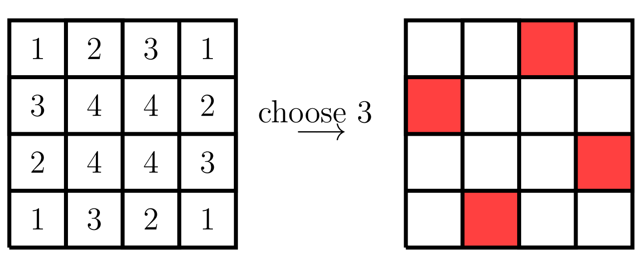


Also, for example, the first two colourings below are equivalent and in Group 2, while the final colouring is in Group 1.

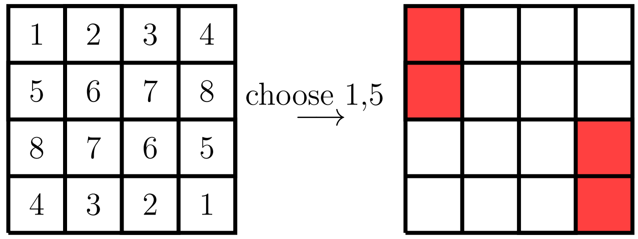


Suppose there are A colourings in Group 1, B colourings in Group 2, and C colourings in Group 3. Since each colouring in Group 1 is equivalent to exactly three other colouring, each colouring in Group 2 is equivalent to exactly one other colouring, and each colouring in Group 3 is equivalent to no other colourings, our final answer will be $A + \frac{B}{2} + \frac{C}{4}$. We now determine each of A, B, and C.

We start with A. We have that $A = 4$, since a colouring in Group 1 may be chosen by choosing either 1, 2, 3, or 4, and then colouring red all squares with the number chosen in the diagram below. (Note every such colouring arises in this way.)



The method for determining B is similar. We obtain a colouring with $180°$ symmetry by choosing two of $1, 2, 3, 4, 5, 6, 7, 8$ and then colouring red all squares with the numbers chosen in the diagram below.



However, four of these choices actually give us colourings in Group 1 (namely, the choices $\{1, 4\}$, $\{2, 8\}$, $\{3, 5\}$, and $\{6, 7\}$). Note every colouring in Group 2 arises from choosing two of the eight numbers. We therefore have that $B = \binom{8}{2} - 4 = 24$.
Finally, we determine C. Since $A + B + C = 1820$, we have that
$C = 1820 - A - B = 1820 - 24 - 4 = 1792$.
Thus our final answer is

$$A + \frac{B}{2} + \frac{C}{4} = 4 + \frac{24}{2} + \frac{1792}{4} = 464.$$

Answer: 464

25. Let $\alpha = \sqrt{1 + \sqrt{1 + x}}$. We then calculate that

$$\frac{32}{\left(1 + \sqrt{1 + \sqrt{1 + x}}\right)^3} + \frac{32}{\left(1 - \sqrt{1 + \sqrt{1 + x}}\right)^3} + \frac{16}{\left(1 + \sqrt{1 + \sqrt{1 + x}}\right)^4} + \frac{16}{\left(1 - \sqrt{1 + \sqrt{1 + x}}\right)^4}$$

$$= \frac{32}{(1 + \alpha)^3} + \frac{32}{(1 - \alpha)^3} + \frac{16}{(1 + \alpha)^4} + \frac{16}{(1 - \alpha)^4}$$

$$= \frac{32[(1 + \alpha)^3 + (1 - \alpha)^3]}{(1 + \alpha)^3(1 - \alpha)^3} + \frac{16[(1 + \alpha)^4 + (1 - \alpha)^4]}{(1 + \alpha)^4(1 - \alpha)^4}$$

$$= \frac{32[2 + 6\alpha^2]}{(1 - \alpha^2)^3} + \frac{32[1 + 6\alpha^2 + \alpha^4]}{(1 - \alpha^2)^4}$$

$$= \frac{32[2 + 4\alpha^2 - 6\alpha^4] + 32[1 + 6\alpha^2 + \alpha^4]}{(1 - \alpha^2)^4}$$

$$= \frac{32[-5\alpha^4 + 10\alpha^2 + 3]}{(1 - \alpha^2)^4}$$

We now address the fact that $\alpha = \sqrt{1 + \sqrt{1 + x}}$. We have that $\alpha^2 = 1 + \sqrt{1 + x}$, $\alpha^4 = 2 + x + 2\sqrt{1 + x}$, and $(1 - \alpha^2)^4 = (\sqrt{1 + x})^4 = (1 + x)^2$. Carrying on from above, we have that

$$\frac{32[-5\alpha^4 + 10\alpha^2 + 3]}{(1 - \alpha^2)^4} = \frac{32[-5(2 + x + 2\sqrt{1 + x}) + 10(1 + \sqrt{1 + x}) + 3]}{(1 + x)^2}$$

$$= \frac{32[3 - 5x]}{(1 + x)^2}$$

Since we are interested in the maximum value of

$$1 - \frac{32}{\left(1 + \sqrt{1 + \sqrt{1 + x}}\right)^3} - \frac{32}{\left(1 - \sqrt{1 + \sqrt{1 + x}}\right)^3} - \frac{16}{\left(1 + \sqrt{1 + \sqrt{1 + x}}\right)^4} - \frac{16}{\left(1 - \sqrt{1 + \sqrt{1 + x}}\right)^4}$$

we are therefore interested in the minimum value of $\dfrac{32[3-5x]}{(1+x)^2}$. The minimum can be found using calculus, but the argument that follows uses more elementary methods. Note that $3-5x = 8-5(1+x)$. Let $u = 1+x$ and let $v = \dfrac{1}{u}$. We then have that

$$\frac{32[3-5x]}{(1+x)^2} = \frac{32(8-5u)}{u^2} = 32(8v^2-5v) = 32v(8v-5)$$

The quadratic function $32v(8v-5)$ has its roots at $v=0$ and $v=\frac{5}{8}$ and opens upwards. Therefore its minimum value occurs at $v=\frac{5}{16}$. This minimum value is

$$32 \times \frac{5}{16} \times (8\frac{5}{16} - 5) = -25.$$

(Note that $v=\frac{5}{16}$ corresponds to $u=\frac{16}{5}$, which corresponds to $x=\frac{11}{5}$, which is indeed in the domain of the original function.) Therefore the maximum of the original function is $1-(-25)=26$.

Answer: 26

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each relay are written as if the value of t is not initially known, and then t is substituted at the end.)

1. (a) Since $x + 3 = 5$ we have that $x = 5 - 3 = 2$. Then $2x + 4 = 2(2) + 4 = 8$.

   (b) The triangle has base $3 - (-5) = 8$ and height $t - 3$, therefore its area is

   $$\frac{8(t-3)}{2} = 4(t-3) = 4(8-3) = 20.$$

   (c) We have that $f(1) = (1 - b)(1 - 3) = 2(b - 1)$. Since $f(1) = t$, we have that $2(b - 1) = t$, so $b = 1 + \frac{t}{2}$. We then have that $f(5) = (5 - b)(2) = (5 - (1 + \frac{t}{2}))(2) = 8 - t = -12$.

   Answer: $(8, 20, -12)$

2. (a) We have that $PR = 32 - 4 = 28$, so $PS = \left(\frac{3}{2}\right) PR = \left(\frac{3}{2}\right) 28 = 42$. Therefore the number located at point S is $4 + 42 = 46$, so $RS = 46 - 32 = 14$. Since $QR = RS$ we also have that $QR = 14$, and so the number located at point Q is equal to $32 - 14 = 18$.

   (b) The x-intercept of a line is the value of x such that $y = 0$. The x-intercept of l_2 occurs when $6x + t = 0$, i.e. when $x = -\frac{t}{6}$. The x-intercept of l_2 occurs when $ax + 24 = 0$, i.e. when $x = -\frac{24}{a}$. Since the two lines have the same x-intercept, we therefore have that $144 = at$.

   The slope-intercept form of l_2 is $y = \left(\frac{a}{2}\right) x + 12$, so l_2 has slope $\frac{a}{2} = \frac{72}{t} = \frac{72}{18} = 4$.

   (c) Squaring both sides yields $4(x - 3)^2 x = t^2 x^2$. Note that $x = 0$ is a solution to this, and that there can be no solution with $x < 0$ due to the presence of $\sqrt{x}$. Cancelling of x from both sides yields $4(x^2 - 6x + 9) = t^2 x$. Grouping together like terms gives us $4x^2 - (24 + t^2)x + 36 = 0$. We now use the quadratic equation: the largest solution of this quadratic equation is

   $$\begin{aligned} x &= \frac{24 + t^2 + \sqrt{(24 + t^2)^2 - 4(4)36}}{8} \\ &= \frac{24 + t^2 + \sqrt{576 + 48t^2 + t^4 - 576}}{8} \\ &= \frac{24 + t^2 + \sqrt{t^2(48 + t^2)}}{8} \\ &= \frac{24 + 16 + \sqrt{16(48 + 16)}}{8} \\ &= \frac{40 + \sqrt{1024}}{8} \\ &= 9 \end{aligned}$$

   Answer: $(18, 4, 9)$

3. (a) We have that $23 + 2x + 61 = 180$, so $2x = 96$, so $x = 48$.

   (b) Suppose Gurpreets initial amount of money is D (units of \$). We are told that $D - \frac{D}{2} - \frac{D}{5} = t$, i.e. that $\frac{3D}{10} = t$, i.e. that $D = \frac{10t}{3} = \frac{10(48)}{3} = 160$.

   (c) Since $5y - x$ and $2y + 5x$ are both widths of the same rectangle, we have that $5y - x = 2y + 5x$, which is equivalent to $y = 2x$.

   The perimeter of the rectangle is $5y - x + 17 + 2y + 5x + 17 = 10x - x + 4x + 5x + 34 = 18x + 34$. That is, $18x + 34 = t$, so $x = \frac{t-34}{18} = \frac{160-34}{18} = 7$

   Answer: $(48, 160, 7)$