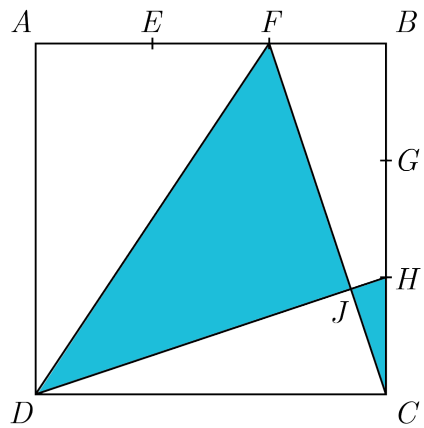
Problem of the Week
Problem E and Solution
A Fraction to Find

Problem

In square $ABCD$, points E and F lie on AB such that $AE = EF = FB = 10$. Similarly, points G and H lie on BC such that $BG = GH = HC = 10$. Let J be the intersection of line segments DH and CF. The areas of $\triangle DFJ$ and $\triangle CJH$ are then shaded.

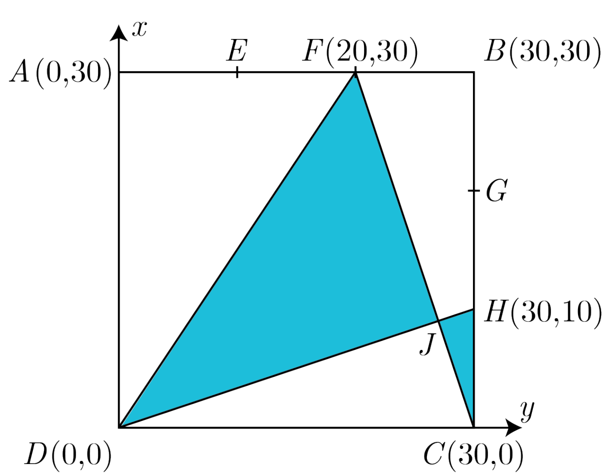


Determine the fraction of the area of square $ABCD$ that is shaded.

Solution

Solution 1

In this solution, we set square $ABCD$ on the Cartesian grid with vertex D at the origin. Then, since $AE = EF = FB = 10$, it follows that $AB = 30$, so the side length of $ABCD$ is 30. Thus, the coordinates of the vertices are $A(0, 30)$, $B(30, 30)$, $C(30, 0)$, and $D(0, 0)$. Since $AF = 20$ and AF is parallel to the x-axis, it follows that the coordinates of F are $(20, 30)$. Since $HC = 10$ and HC is parallel to the y-axis, it follows that the coordinates of H are $(30, 10)$.



Now we will find the coordinates of J by finding the point of intersection of the lines through DH and CF.

The line through DH has slope $\frac{10}{30} = \frac{1}{3}$ and y-intercept 0. Thus, its equation is $y = \frac{1}{3}x$.

The line through CF has slope $\frac{0-30}{30-20} = \frac{-30}{10} = -3$. To determine its y-intercept we substitute $(30, 0)$ into $y = -3x + b$. This gives $0 = -3(30) + b$, so $b = 90$. Thus, its equation is $y = -3x + 90$.

We set these two equations equal to each other to solve for the point of intersection.

$$\begin{aligned}\frac{1}{3}x &= -3x + 90\\ \frac{10}{3}x &= 90\\ x &= 90 \times \frac{3}{10} = 27\end{aligned}$$

Thus, $y = \frac{1}{3}(27) = 9$, so the coordinates of J are $(27, 9)$.

Since the slope of the line through DH is $\frac{1}{3}$ and the slope of the line through CF is -3, and these are negative reciprocals, it follows that these lines are perpendicular. Thus, both $\triangle DFJ$ and $\triangle CJH$ are right-angled triangles. So to determine the area of $\triangle DFJ$ we will consider the base to be DJ and the height to be JF. Using the Pythagorean theorem,

$$\begin{aligned}DJ &= \sqrt{27^2 + 9^2}\\ &= \sqrt{810}\\ &= 9\sqrt{10}\end{aligned}$$

$$\begin{aligned}JF &= \sqrt{(27-20)^2 + (9-30)^2}\\ &= \sqrt{7^2 + (-21)^2}\\ &= \sqrt{490} = 7\sqrt{10}\end{aligned}$$

Thus,

$$\text{Area } \triangle DFJ = \frac{1}{2} \times DJ \times JF = \frac{1}{2} \times 9\sqrt{10} \times 7\sqrt{10} = \frac{1}{2} \times 63 \times 10 = 315$$

To determine the area of $\triangle CJH$ we will consider the base to be $CH = 10$. To find the height, we drop a perpendicular from J to CH and call this point K. Then the height is $JK = 30 - 27 = 3$. Thus,

$$\text{Area } \triangle CJH = \frac{1}{2} \times CH \times JK = \frac{1}{2} \times 10 \times 3 = 15$$

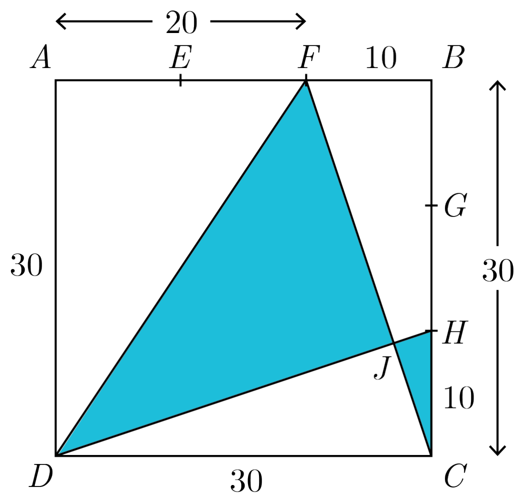
Then the total area shaded is $315 + 15 = 330$.

The area of square $ABCD$ is $30 \times 30 = 900$. Therefore, the fraction of this area that is shaded is $\frac{330}{900} = \frac{11}{30}$.

Solution 2

In this solution we use similar triangles to calculate the area indirectly.

Since $AE = EF = FB = 10$, and $ABCD$ is a square, it follows that $AB = BC = CD = AD = 30$. Since $AE = EF = 10$ it follows that $AF = 20$.



We can then determine the areas of right-angled triangles $\triangle DAF$ and $\triangle CBF$. The area of $\triangle DAF$ is $\frac{1}{2} \times 30 \times 20 = 300$ and the area of $\triangle CBF$ is $\frac{1}{2} \times 30 \times 10 = 150$.

Next we determine the area of $\triangle CJH$. First we notice that $\triangle CBF$ and $\triangle DCH$ are congruent, and $\triangle DCH$ has been rotated by 90°. Thus, $DH \perp CF$. Next we notice that $\angle HCJ = \angle BCF$ and $\angle CJH = \angle CBF = 90^\circ$. Thus, $\triangle CJH \sim \triangle CBF$.

Using the Pythagorean theorem, $CF = \sqrt{30^2 + 10^2} = \sqrt{1000} = 10\sqrt{10}$. Since the corresponding side in $\triangle CJH$ is $CH = 10$, it follows that the side lengths in $\triangle CBF$ are $\sqrt{10}$ times the length of their corresponding sides in $\triangle CJH$. Thus, the area of $\triangle CBF$ is $\sqrt{10} \times \sqrt{10} = 10$ times the area of $\triangle CJH$. Since the area of $\triangle CBF$ is 150, it follows that the area of $\triangle CJH$ is 15.

Finally, we can determine the total area shaded.

$$\begin{aligned}
\text{Shaded area} &= \text{Area } ABCD - \text{Area } \triangle DAF - 2 \times \text{Area } \triangle CBF + 2 \times \text{Area } \triangle CJH \\
&= 30 \times 30 - 300 - 2 \times 150 + 2 \times 15 \\
&= 330
\end{aligned}$$

Therefore, the fraction of the area of square $ABCD$ that is shaded is $\frac{330}{900} = \frac{11}{30}$.