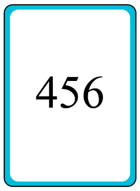

Problem of the Week

Problem E and Solution

A New Deck of Cards

Problem

Joanna has a deck of cards. Each card in the deck has a three-digit positive integer on it, and there is exactly one card in the deck for every three-digit positive integer.

Joanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.

Solution

To begin, we determine the number of cards in the deck. There are 999 positive integers less than 1000. Of these, 90 are two-digit integers and 9 are single-digit integers. Therefore, there are $999 - 90 - 9 = 900$ three-digit positive integers, and so 900 cards in the deck.

Next, we determine the digit combinations that sum to 15.

- **Case 1:** One of the digits on the card is a 0.
  Then the other two digits on the card must add to 15. There are two possibilities for the digits. The three digits must be $0, 6, 9$ or $0, 7, 8$.

- **Case 2:** One of the digits on the card is a 1, but the number does not contain a 0.
  Then the other two digits on the card must add to 14. There are three possibilities for the digits. The digits must be $1, 5, 9$ or $1, 6, 8$ or $1, 7, 7$.

- **Case 3:** One of the digits on the card is a 2, but the number does not contain a 0 or 1.
  Then the other two digits on the card must add to 13. There are three possibilities for the digits. The digits must be $2, 4, 9$ or $2, 5, 8$ or $2, 6, 7$.

- **Case 4:** One of the digits on the card is a 3, but the number does not contain a 0, 1, or 2.
  Then the other two digits on the card must add to 12. There are four possibilities for the digits. The digits must be $3, 3, 9$ or $3, 4, 8$ or $3, 5, 7$ or $3, 6, 6$.

- **Case 5:** One of the digits on the card is a 4, but the number does not contain a 0, 1, 2, or 3.
  Then the other two digits on the card must add to 11. There are two possibilities for the digits. The digits must be $4, 4, 7$ or $4, 5, 6$.

- **Case 6:** One of the digits on the card is a 5, but the number does not contain a 0, 1, 2, 3, or 4.
  Then the other two digits on the card must add to 10. There is only one possibility for the digits. The digits must be $5, 5, 5$.

Now that we know what combinations of digits can be on the cards, we can determine the number of cards that can be created from each combination.

- **Case (a):** One of the digits on the card is 0.
  Earlier we found that there were two such digit combinations: $0, 6, 9$ and $0, 7, 8$. This is a special case since 0 cannot appear in the number as the hundreds digit for the number to be a three-digit number. For each of the two digit combinations, the 0 can be placed in two ways, in the tens digit or the units digit. Once the 0 is placed, the other two numbers can be placed in the remaining two spots in two ways. Thus, each digit combination can form $2 \times 2 = 4$ three-digit numbers. Since there are two digit combinations, there are $2 \times 4 = 8$ cards in the deck that contain a 0 whose digits add to 15.

- **Case (b):** All three digits on the card are different and the number does not contain a 0.
  From the earlier cases, there are eight such combinations: $1, 5, 9$, and $1, 6, 8$, and $2, 4, 9$, and $2, 5, 8$, and $2, 6, 7$, and $3, 4, 8$, and $3, 5, 7$, and $4, 5, 6$. For each of these combinations, the hundreds digit can be placed in three ways. For each of these three choices for hundreds digit, the tens digit can be placed in two ways. Once the hundreds digit and tens digit are selected, the units digit must get the third number. So each combination can form $3 \times 2 = 6$ different numbers. Since there are eight digit combinations, there are $8 \times 6 = 48$ cards in the deck that contain three different digits other than 0 whose digits add to 15.

- **Case (c):** Two of the digits on the card are the same and the number does not contain a 0.
  From the earlier cases, there are four such combinations: $1, 7, 7$, and $3, 3, 9$, and $3, 6, 6$, and $4, 4, 7$. For each of these combinations, the unique number can be placed in one of three spots. Once the unique number is placed the other two numbers must go in the remaining two spots. So each digit combination can form three different numbers. Since there are four digit combinations, there are $4 \times 3 = 12$ cards in the deck that do not contain a 0 but contain two digits the same and whose digits add to 15.

- **Case (d):** The three digits on the card are the same.
  From the earlier cases we discovered only one such combination: $5, 5, 5$. Only one card can be produced using the numbers from this combination.

Combining the counts from the above four cases, there are $8 + 48 + 12 + 1 = 69$ cards in the deck with a digit sum of 15. Therefore, the probability that Joanna selects card whose digits add to 15 is $\frac{69}{900} = \frac{23}{300}$. This translates to approximately a 7.7% chance.

A game is considered *fair* if there is close to a 50% chance of winning. If Joanna was playing a game where she can win by drawing a card whose digits sum to 15, then this game is definitely not fair.

Can you create a game using this specific deck of cards that is reasonably fair and fun to play?