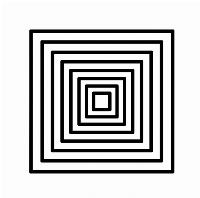

Problem of the Week

Problem E and Solution

Nested Squares

Problem

The prime factorization of 20 is $2^2 \times 5$.

The number 20 has 6 positive divisors. They are:

$$2^0 5^0 = 1,\ 2^0 5^1 = 5,\ 2^1 5^0 = 2,\ 2^1 5^1 = 10,\ 2^2 5^0 = 4,\ \ 2^2 5^1 = 20$$

Two of the divisors, 1 and 4, are perfect squares.

How many positive divisors of 2025^{2025} are perfect squares?

Solution

First, let's look at the prime factorization of four different perfect squares:

$$9 = 3^2,\ 16 = 2^4,\ 36 = 2^2 \times 3^2,\ \ 129\,600 = 2^6 \times 3^4 \times 5^2$$

Note that, in each case, the exponent on each of the prime factors is even. In fact, a positive integer is a perfect square exactly when the exponent on each of the prime factors in its prime factorization is an even integer greater than or equal to zero. Now

$$\begin{aligned} 2025^{2025} &= (3^4 \times 5^2)^{2025} \\ &= (3^4)^{2025} \times (5^2)^{2025} \\ &= 3^{8100} \times 5^{4050} \end{aligned}$$

All positive divisors of 2025^{2025} will be of the form $3^k \times 5^n$, $0 \leq k \leq 8100$, and $0 \leq n \leq 4050$, where k and n are each integers.

For $3^k \times 5^n$ to be a perfect square, k must be an even integer such that $0 \leq k \leq 8100$ and n must be an even integer such that $0 \leq n \leq 4050$.

There are $8100 \div 2 = 4050$ even integers from 1 to 8100. The number 0 is also even, so there are 4051 possible values of k.

There are 2025 even integers from 1 to 4050. The number 0 is also even, so there are 2026 possible values of n.

For each of the 4051 values of k, there are 2026 values of n so there are $4051 \times 2026 = 8\,207\,326$ perfect square divisors of 2025^{2025}.

Therefore, 2025^{2025} has 8 207 326 positive divisors that are perfect squares. This is over 8 million perfect square divisors!