Problem of the Week

Problem E and Solution

Power Trail

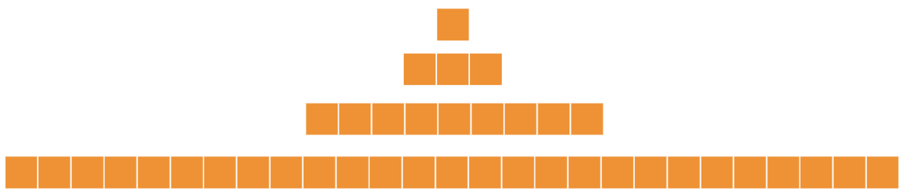
Problem

The first four terms of a geometric sequence are

$$a,\ b,\ 4a,\ a-6+b$$

for some numbers a and b with $a \neq 0$.

Determine all possibilities for the values of a and b.



NOTE: A geometric sequence is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 2, 6, 18 is a geometric sequence with three terms and a common ratio of 3. You can explore geometric sequences further in our CEMC Courseware.

Solution

In a geometric sequence with first term t_1, second term t_2, third term t_3, and fourth term t_4, we have $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3} = \dfrac{t_3}{t_4}$.

Here, $t_1 = a$, $t_2 = b$, $t_3 = 4a$, and $t_4 = a-6+b$.

Therefore, from $\dfrac{t_1}{t_2} = \dfrac{t_2}{t_3}$, we have

$$\begin{aligned} \frac{a}{b} &= \frac{b}{4a} \\ 4a^2 &= b^2 \\ 2a &= \pm b \end{aligned}$$

There are now two cases to consider.

- **Case 1:** When $b = 2a$, then the sequence can be written as a, $2a$, $4a$, $3a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned} \frac{a}{2a} &= \frac{4a}{3a-6} \\ 3a^2 - 6a &= 8a^2 \\ 0 &= 5a^2 + 6a \\ &= a(5a+6) \end{aligned}$$

  Thus, $a = 0$ or $a = -\frac{6}{5}$. Since $a \neq 0$, we have $a = -\frac{6}{5}$.
  Then $b = 2a = 2(-\frac{6}{5}) = -\frac{12}{5}$ and $a - 6 + b = -\frac{6}{5} - 6 + (-\frac{12}{5}) = -\frac{48}{5}$
  Therefore, the sequence, a, b, $4a$, $a - 6 + b$ is $-\frac{6}{5}$, $-\frac{12}{5}$, $-\frac{24}{5}$, $-\frac{48}{5}$.
  This sequence is indeed geometric, with first term $-\frac{6}{5}$ and common ratio 2.

- **Case 2:** When $b = -2a$, then the sequence can be written as a, $-2a$, $4a$, $-a - 6$.

  Then the ratio $\dfrac{t_1}{t_2} = \dfrac{t_3}{t_4}$ simplifies to

$$\begin{aligned} \frac{a}{-2a} &= \frac{4a}{-a-6} \\ -a^2 - 6a &= -8a^2 \\ 7a^2 - 6a &= 0 \\ a(7a-6) &= 0 \end{aligned}$$

  Thus, $a = 0$ or $a = \frac{6}{7}$. Since $a \neq 0$, we have $a = \frac{6}{7}$.
  Then $b = -2a = -2(\frac{6}{7}) = -\frac{12}{7}$ and $a - 6 + b = \frac{6}{7} - 6 + (-\frac{12}{7}) = -\frac{48}{7}$
  Therefore, the sequence, $a, b, 4a, a - 6 + b$ is $\frac{6}{7}$, $-\frac{12}{7}$, $\frac{24}{7}$, $-\frac{48}{7}$.
  This sequence is indeed geometric, with first term $\frac{6}{7}$ and common ratio -2.

Therefore, there are two possibilities. It could be the case that $a = -\frac{6}{5}$ and $b = -\frac{12}{5}$, or it could be the case that $a = \frac{6}{7}$ and $b = -\frac{12}{7}$.