$w^x + y^z$

Problem of the Week
Problem E and Solution
To the Peak with Powers

Problem

Avital can choose four different numbers w, x, y, and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

Solution

Consider w^x, where w and x are different numbers from $\{-1, -2, -3, -4, -5\}$. What is the largest possible value for w^x?

Since x will be negative, we write $w^x = \frac{1}{w^{-x}}$, where $-x > 0$.

If x is odd, then since w is negative, then w^x will be negative.
If x is even, then w^x will be positive.

So to make w^x as large as possible, we make x even. So $x = -2$ or $x = -4$.

Also, in order to make $w^x = \frac{1}{w^{-x}}$ as large as possible, we want to make the denominator, w^{-x}, as small as possible, so w should be as small as possible in absolute value.

Therefore, the largest possible value of w^x will be when $w = -1$ and $x = -2$ or $x = -4$. In both cases, $w^x = 1$, since $(-1)^{-2} = (-1)^{-4} = 1$.

What is the second largest possible value for w^x?

Again, we need x to be even to make w^x positive, and from above, we can assume that $w \neq -1$.

When $x = -2$, the smallest possible base (in absolute value) is $w = -3$ and $w^x = \frac{1}{(-3)^2} = \frac{1}{9}$.

When $x = -4$, the smallest possible base (in absolute value) is $w = -2$ and $w^x = \frac{1}{(-2)^4} = \frac{1}{16}$.

The largest of these two values for w^x is $\frac{1}{9}$.

Therefore, the two largest possible values for w^x are 1 and $\frac{1}{9}$.

Thus, the largest possible value of $w^x + y^z$ is $1 + \frac{1}{9} = \frac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$. This will occur when $w = -1$, $x = -4$, $y = -3$, and $z = -2$ or when $w = -3$, $x = -2$, $y = -1$, and $z = -4$.